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LINEAR RISK-AVERSE OPTIMAL CONTROL PROBLEMS: APPLICATIONS IN ECONOMICS AND FINANCE

Paolo Vitale

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Arcelli Centre for Monetary and Financial Studies
Department of Economics and Business
LUISS Guido Carli
Viale Romania 32, 00197, Rome -- Italy
http://casmef.luiss.edu

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Linear Risk-averse Optimal Control Problems: Applications in Economics and Finance

Paolo Vitale
*University of Pescara**

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ABSTRACT

We discuss how Whittle's (Whittle, 1990) approach to risk-sensitive optimal control problems can be applied in economics and finance. We show how his analysis of the class of Linear Exponential Quadratic Gaussian problems can be extended to accommodate time-discounting, while preserving its simple and general recursive solutions. We apply Whittle's methodology investigating two specific problems in financial economics and monetary policy.

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*Department of Economics, Università d'Annunzio, Viale Pindaro 42, 65127 Pescara (Italy); telephone ++39-085-453-7647; webpage: `http:/www.unich.it/~vitale`; e-mail: `p.vitale@unich.it`

Introduction

Risk-aversion is an important aspect of agents' preferences, which heavily influences their actions, in particular when the economic environment is complex and uncertain, and agents need to consider the long-run implications of their decisions.

Under risk-neutrality optimal control problems can be easily solved employing well-established results. Thus, in solving a linear quadratic problem straightforward recursive formulae immediately yield the optimal control rule, while applying the certainty equivalence principle unknown variables can be replaced by their maximum likelihood estimates.

Whittle's (Whittle, 1990) shows how a modified (or risk-sensitive) certainty equivalence principle and recursive formulae can be derived when risk-aversion is introduced in the standard linear quadratic set-up. We present his formulation of the *linear exponential quadratic Gaussian* optimal control problem which allows to achieve such a goal. We show how his methodology can be easily employed in economics and finance discussing two specific problems.[1]

Whittle's methodology is derived within a finite horizon framework and does not consider time-discounting. Both are common features of many economic problems. Risk-sensitive optimal control problems with time-discounting have however been discussed by Hansen and Sargent (Hansen and Sargent, 1994, 2005). They propose a recursive minimization criterion *à la* Epstein and Zin which allows to combine risk-aversion and time-discounting. We modify their minimization criterion and show how the recursive solution method proposed by Whittle can be extended to risk-sensitive optimal control problems with time-discounting and infinite horizon. We are then able to reformulate Whittle's risk-sensitive certainty equivalence principle and recursive formulae for the optimal control rule to accommodate time-discounting and infinite horizon. Following Whittle's lead, we are also able to analyze risk-sensitive optimal control problems with time-discounting under imperfect state observation, a scenario that Hansen and Sargent do not explicitly consider and that their recursive minimization criterion cannot accommodate.

This paper is organized as follows. In Section 1 we present Whittle's linear exponential quadratic Gaussian problem. This is an extension of the standard linear quadratic problem that allows to accommodate risk-aversion. The risk-sensitive certainty equivalence principle is discussed, alongside the risk-sensitive Riccati equation, which yields the optimal control rule

[1] Mamaysky and Spiegel (2002), Van der Ploeg (2003,2007), and Zhang (2004) employ Whittle's methodology to investigate specific problems in economics and finance.

for the class of Markovian linear exponential quadratic Gaussian problems. The risk-sensitive separation principle is also presented: this allows to separate control and estimation in the case of imperfect state observation. In the case of imperfect observation a Risk-sensitive Kalman filter applies.

In Section 2 we apply Whittle's methodology to an important problem from financial economics. Specifically we extend Kyle's (Kyle, 1985) analysis of a sequential auction market to the case in which the insider is risk-averse and private information pertains to a collection of risky assets. This allows to see how Whittle's methodology can be employed within a game setting to derive a perfect Bayesian equilibrium. With our multi-asset formulation, we confirm Holden and Subrahmanyam's (Holden and Subrahmanyam, 1994) apparently counter-intuitive result that risk-aversion induces informed agents to consume more rapidly their long-lived private information *vis-a-vis* their risk-neutral counter-parts, so that with risk-averse insiders securities markets are more efficient. In addition, we extend to a multi-period formulation Caballè and Krishnan's (Caballè and Krishnan, 1994) conclusion that in equilibrium the insider operates in such a way that the price impact of signed volume (order flow) is symmetric across all risky assets.

In the following Section we introduce time-discounting to the class of linear exponential quadratic Gaussian problems, proposing a recursive minimization criterion which differs from that put forward by Hansen and Sargent. The suggested recursive criterion allows: i) to apply, with simple adjustments, Whittle's methodology and derive recursive solution formulae for the optimal control rule; and ii) to solve LEQG problems with time-discounting when only noisy signals on the state variables are observed, a scenario which cannot be investigated using Hansen and Sargent's recursive criterion.

In Section 4 we apply the linear exponential quadratic Gaussian framework with time-discounting to the problem of output and inflation stabilization on the part of an independent central bank investigated by Svensson (Svennson, 1997). We extend his analysis to the case in which the central bank is risk-averse.[2] Because of risk-aversion the standard certainty equivalence principle cannot be applied and hence: i) the inflation forecast is not longer an explicit intermediate target when inflation targeting is the exclusive mission of the central bank; and ii) the monetary authorities do not necessarily expect the inflation rate to mean revert to its target level when the monetary policy is also aimed at output stabilization. We

[2] van der Ploeg (2004) investigates a similar extension of Svensson's analysis. However, he introduces neither time-discount nor a recursive minimization criterion.

actually see that even without output stabilization, a scenario in which the inflation forecast is always equal to its target level under risk-neutrality, if the central bank is risk-averse it may well be that the monetary authorities expect the inflation rate to wander away from the target level. In addition, we find that under risk-aversion the central bank follows a more aggressive Taylor rule. Finally, we investigate the possibility that the central bank observes the state variables with a time lag so as to employ within this context Whittle's risk-sensitive separation principle.

1 Linear Exponential Quadratic Optimal Control Problem

Let us define a specific class of optimal control problems, generally referred as Linear Exponential Quadratic Gaussian, in the formulation proposed by Whittle (Whittle, 1990).[3]

Definition 1 (LEQG) *An optimal control problem is said to be Linear Exponential Quadratic Gaussian if the following criterion*

$$\ln\left(E\left[\exp\left(\rho\frac{\boldsymbol{\mathcal{C}}}{2}\right)\right]\right), \tag{1.1}$$

where $\rho > 0$, is minimized over T (with T finite) periods with respect to the control variables $\mathbf{u}_t$ $(t = 1, \ldots, T-1)$, under the following conditions:
(i) in any period t, $\mathbf{u}_t$ can take any value in some finite-dimensional vector space;
(ii) $\boldsymbol{\mathcal{C}}$ is a scalar-valued cost function that can be expressed in the following form

$$\boldsymbol{\mathcal{C}} \;=\; Q(\mathbf{U}_{T-1}, \boldsymbol{\xi}),$$

where $\mathbf{U}_{T-1} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{T-1})$ is the complete control-path and $\boldsymbol{\xi}$ is a vector-valued noise term;
(iii) Q is quadratic in all the arguments and positive definite in $\mathbf{U}_{T-1}$;
(iv) the unconditional distribution of $\boldsymbol{\xi}$ is multi-normal with vector of means $\mathbf{0}$ and covariance matrix $\boldsymbol{\Upsilon}$ independent of the control vector;
(v) the observable vector, $\mathbf{w}_t$, is reducible to linear functions of $\boldsymbol{\xi}$.

[3] As we only allow for positive values of the risk-sensitive coefficient ρ we restrict our discussion to the *risk-averse* specification. See Whittle (1990) for the analysis of the complementary *risk-seeking* specification (with $\rho < 0$).

Imposing the condition that the optimal problem is solved over a finite horizon T ensures that the criterion is well defined. In Section 3 we will discuss how, introducing time-discounting, an infinite horizon can be accommodated. Condition *(iii)* that the quadratic function Q is positive definite in the control vector guaranties that the minimum of the criterion is determined via first order conditions. However, as shown in the examples we will discuss in Sections 2 and 4, while sufficient such condition is not necessary. Similarly, condition *(ii)* that the control vector is free-valued, and hence not subject to any constraint, is not required for the existence of a minimum of the criterion and it could be disposed of. Nevertheless, it is extremely useful in characterizing the optimal control path, in that it allows to derive recursive solutions for the Markovian version of the LEGQ problem.

Expressing the cost function $\boldsymbol{\mathcal{C}}$ in reduced-form simplifies the derivation of the risk-sensitive certainty equivalence and separation principles obtained by Whittle. Such reduced-form is however perfectly consistent with the standard (extensive-form) formulation where the cost function depends on the state and control path (respectively $\mathbf{Z}_T = (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_T)$ and $\mathbf{U}_{T-1}$) we will consider when analyzing the Markovian LEQG problem. Similarly, while in condition *(v)* the vector $\mathbf{w}_t$ is expressed in reduced-form, in the Markovian LEQG problem it will be written in the standard formulation as a function of the state variable $\mathbf{z}_t$.

Under the conditions of Definition 1, let $\boldsymbol{\mathcal{S}}$ denote the *total stress* function. This is defined as follows

$$\boldsymbol{\mathcal{S}} \;=\; \boldsymbol{\mathcal{C}} \;-\; \frac{1}{\rho}\,\boldsymbol{\mathcal{D}}\,,$$

where $\boldsymbol{\mathcal{D}} = \boldsymbol{\xi}^{'}\boldsymbol{\Upsilon}^{-1}\boldsymbol{\xi}$ is referred to as the *discrepancy* function. Such function appears in the calculation of the expectation in the risk-sensitive optimizazion criterion, $\ln(E[\exp(\rho\boldsymbol{\mathcal{C}}/2)])$. Then, one can obtain the following Theorem, that defines a modified Risk-sensitive Certainty Equivalence Principle (Risk-sensitive CEP) for the class of LEQG problems.

Theorem 1 - *(Risk-sensitive Certainty Equivalence Principle). The optimal value of the vector* $\mathbf{u}_t$ *is determined by simultaneously minimizing* $\boldsymbol{\mathcal{S}}$ *with respect to* $\mathbf{u}_t, \mathbf{u}_{t+1}, \ldots, \mathbf{u}_T$ *and maximizing it with respect to the unobservable* $\mathbf{w}_{t+1}, \mathbf{w}_{t+2}, \ldots, \mathbf{w}_T$*. In other words, an optimal current decision is obtained by minimizing with respect to all decision currently unmade (future values of the control vector* $\mathbf{u}_t$*) and maximizing with respect to all the quantity currently unobservable (future values of the observable vector* $\mathbf{w}_t$*).*

While details of the proof of this and the following three Theorems are given in Whittle (1990), we can discuss the main intuition. The result is based on the properties of the integral of the exponential of a quadratic form. In fact, if $Q(\mathbf{z}, \mathbf{u})$ is a quadratic form which is positive definite in $\mathbf{z}$ and negative definite in $\mathbf{u}$ the following holds

$$\min_{\mathbf{u}} \int \exp\left[-\frac{1}{2} Q(\mathbf{z}, \mathbf{u})\, d\mathbf{z}\right] \propto \exp\left[-\frac{1}{2} \max_{\mathbf{u}} \min_{\mathbf{z}} Q(\mathbf{z}, \mathbf{u})\right],$$

where the proportionality constant does not depend on $\mathbf{u}$. Then, in the context of the minimization of the risk-sensitive criterion, it is shown that the following holds

$$\min_{\mathbf{u}} E\left[\exp\left(\rho \frac{\boldsymbol{\mathcal{C}}}{2}\right)\right] \propto \min_{\mathbf{u}} \int \exp\left(\frac{1}{2}(\rho \boldsymbol{\mathcal{C}} - \boldsymbol{\mathcal{D}})\right) d\boldsymbol{\xi} = \min_{\mathbf{u}} \int \exp\left(\rho \frac{\boldsymbol{\mathcal{S}}}{2}\right) d\boldsymbol{\xi}.$$

Applying the aforementioned result, we find that the following holds

$$\min_{\mathbf{u}} E\left[\exp\left(\rho \frac{\boldsymbol{\mathcal{C}}}{2}\right)\right] \propto \exp\left(\frac{\rho}{2} \min_{\mathbf{u}} \max_{\boldsymbol{\xi}} \boldsymbol{\mathcal{S}}\right).$$

This shows that to minimize the optimization criterion is sufficient to satisfy a *saddle-point* condition defined on the total stress function: $\boldsymbol{\mathcal{S}}$ is first maximized with respect to the noise vector $\boldsymbol{\xi}$ and then minimized with respect to the control vector $\mathbf{u}$.[4] Using this result recursively the Theorem is proved.

This Theorem extends the Certainty Equivalence Principle of the Linear Quadratic Gaussian (LQG) problem: the normally distributed unobservable variables are no longer replaced by their estimates, but by those that maximize the total stress in order to compensate for risk-aversion. In other words, in the LQG optimal control problem the separation principle between optimization of the control vector and estimation of the unknown values applies, in that the control vector is chosen by minimizing the criterion as it would be in the perfect information case, with the unobservable values replaced by their maximum likelihood (ML) estimates. On the contrary, in the risk-sensitive case the derivation of the optimal control vector and the optimal estimation of the unknown values are intertwined, as the optimal control and optimal estimates are chosen in order to *extremize* the total stress function. Indeed, differently from the LQG problem, uncertainty over the noise vector conditions the optimal choice of the control vector. Specifically, the statistical characteristics of $\boldsymbol{\xi}$, and hence its covariance matrix $\boldsymbol{\Upsilon}$, influence the optimal value of the vector $\mathbf{u}_t$. *Viceversa*, the shape of the cost function affects

[4] Maximizing $\boldsymbol{\mathcal{S}}$ with respect to $\boldsymbol{\xi}$ and minimizing it with respect to $\mathbf{u}$ is referred to as the *extremization* of the total stress function.

the optimal estimate of the unknown values, which no longer corresponds to ML estimate but it is given by the *maximum total stress* estimate (MTSE).

Theorem 1 indicates that closed-loop control rules can be determined through the saddle point condition for the total stress function. This defines the optimal control vector whatever values previous variables may have taken ($\mathbf{u}_t(\mathbf{H}_t)$ where $\mathbf{H}_t = \{\mathbf{h}_0, \mathbf{U}_{t-1}, \mathbf{W}_t\}$ is observation history, with $\mathbf{W}_t = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_t)$, and $\mathbf{h}_0$ is initial information). This Risk-sensitive CEP is particularly useful when we consider a Markovian LEQG problem. For this sub-class of LEQG problems it is possible to operate by backward induction and derive a Risk-sensitive Separation Principle (Risk-sensitive SP) between estimation and control. Applying this Risk-sensitive SP is then possible to define straightforward recursive control rules and estimation formulae.

Definition 2 *A LEQG problem is Markovian if it satisfies the following conditions:*
(i) the vector of state variables, $\mathbf{z}_t$, is governed by the following liner plant equation

$$\mathbf{z}_t \;=\; \mathbf{A}\,\mathbf{z}_{t-1} + \mathbf{B}\,\mathbf{u}_{t-1} + \boldsymbol{\epsilon}_t\,;$$

(ii) the vector of observable variables is given by

$$\mathbf{w}_t \;=\; \mathbf{C}\,\mathbf{z}_{t-1} + \boldsymbol{\eta}_t\,,$$

with $\boldsymbol{\psi}_t = \begin{pmatrix} \boldsymbol{\epsilon}_t \\ \boldsymbol{\eta}_t \end{pmatrix} \sim \boldsymbol{\mathcal{N}}\left[\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{N} & \mathbf{L} \\ \mathbf{L}' & \mathbf{M} \end{pmatrix}\right]$;
(iii) the cost function $\boldsymbol{\mathcal{C}}$ is decomposable in the following way

$$\boldsymbol{\mathcal{C}} \;=\; \sum_{t=1}^{T-1} c_t + \boldsymbol{\mathcal{C}}_T$$

where c_t is a quadratic function in the control and the state variables $(\mathbf{u}_t, \mathbf{z}_t)$, with

$$c_t \;=\; \begin{pmatrix} \mathbf{z}_t' & \mathbf{u}_t' \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{S}' \\ \mathbf{S} & \mathbf{Q} \end{pmatrix} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{u}_t \end{pmatrix} \quad \textit{and} \quad \boldsymbol{\mathcal{C}}_T \;=\; \mathbf{z}_T'\,\boldsymbol{\Pi}\,\mathbf{z}_T\,,$$

In Definition 2 the LEQG problem is time-homogeneous, in that neither the plant equation nor the cost function explicitly depends on time t. However, this Definition can be adjusted to accommodate a non-homogenous plant equation and/or cost function, by making any of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ time-dependent. Definition 2 also entails imperfect

observation of the state vector $\mathbf{z}_t$, in that at time t only a vector of noisy signals on time $t-1$ state vector is observed, $\mathbf{w}_t = \mathbf{C}\mathbf{z}_{t-1} + \boldsymbol{\eta}_t$. Even this hypothesis can be modified or relaxed, so that in the perfect state observation case the current state vector is always observable $\mathbf{w}_t = \mathbf{z}_t$.

Given Definition 2 the discrepancy component of the total stress is equal to

$$\boldsymbol{\mathcal{D}} \quad = \quad \boldsymbol{\mathcal{D}}_0 + \sum_{t=1}^{T} d_t \,,$$

where d_t is a quadratic form in $\boldsymbol{\psi}_t$ and $\boldsymbol{\mathcal{D}}_0$ is a quadratic form in the vector $\mathbf{z}_0$ and its expectation (conditional on the initial information $\mathbf{h}_0$) $\hat{\mathbf{z}}_0$, with

$$d_t \;=\; \left(\boldsymbol{\epsilon}_t' \;\boldsymbol{\eta}_t'\right) \begin{pmatrix} \mathbf{N} & \mathbf{L}' \\ \mathbf{L} & \mathbf{M} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\epsilon}_t \\ \boldsymbol{\eta}_t \end{pmatrix}, \qquad \boldsymbol{\mathcal{D}}_0 \;=\; (\mathbf{z}_0 - \hat{\mathbf{z}}_0)' \, \boldsymbol{\Omega}^{-1} \, (\mathbf{z}_0 - \hat{\mathbf{z}}_0)$$

and $\boldsymbol{\Omega}$ the covariance matrix for $\mathbf{z}_0$ conditional on $\mathbf{h}_0$. Hence, the total stress function can be written as

$$\boldsymbol{\mathcal{S}} \;=\; \sum_{t=1}^{T-1} c_t \;+\; \boldsymbol{\mathcal{C}}_T \;-\; \frac{1}{\rho} \left(\boldsymbol{\mathcal{D}}_0 \;+\; \sum_{t=1}^{T} d_t \right),$$

an expression which under perfect state observation collapses to

$$\boldsymbol{\mathcal{S}} \;=\; \sum_{t=1}^{T-1} c_t \;+\; \boldsymbol{\mathcal{C}}_T \;-\; \frac{1}{\rho} \left(\boldsymbol{\mathcal{D}}_0 \;+\; \sum_{t=1}^{T} \boldsymbol{\epsilon}_t' \mathbf{N}^{-1} \boldsymbol{\epsilon}_t \right).$$

These expressions clearly indicate that the total stress function is time-separable. Then, in applying the Risk-sensitive CEP at time t, the *etremization* of the total stress function can be achieved by splitting $\boldsymbol{\mathcal{S}}$ into two components, denoted as *past stress* and *future stress*, which contain terms allocated respectively to past and future. Then, the following Definition applies.

Definition 3 *At time t the extremized past stress and future stress, $P_t(\mathbf{z}_t, \mathbf{H}_t)$ and $F_t(\mathbf{z}_t)$, are given by the following expressions*

$$P_t(\mathbf{z}_t, \mathbf{H}_t) \quad = \quad \max_{\mathbf{z}_0, \ldots, \mathbf{z}_{t-1}} \left[\sum_{h=1}^{t-1} c_h \;-\; \frac{1}{\rho} \left(\boldsymbol{\mathcal{D}}_0 \;+\; \sum_{h=1}^{t} d_h \right) \right],$$

$$F_t(\mathbf{z}_t) \quad = \quad \min_{\mathbf{u}_t, \ldots, \mathbf{u}_{T-1}} \left\{ \max_{\substack{\mathbf{z}_{t+1}, \ldots, \mathbf{z}_T \\ \mathbf{w}_{t+1}, \ldots, \mathbf{w}_T}} \left[\sum_{h=t}^{T-1} c_h \;+\; \boldsymbol{\mathcal{C}}_T \;-\; \frac{1}{\rho} \sum_{h=t+1}^{T} d_h \right] \right\}.$$

According to this Definition the extremized past stress and future stress are obtained by minimizing with respect to the future values of the control vector and maximizing with respect to the past and future values of the state variables. Only the current state vector $\mathbf{z}_t$ appears in both expressions and it is left undetermined. By holding $\mathbf{z}_t$ free at any time t, it is possible to separate the problem of the stress extremization between past and future. The following Theorem, which suggests a simple method to find the optimal control at any time t and introduces a Risk-sensitive Separation Principle (Risk-sensitive SP) for the Markovian LEQG problem.

Theorem 2 - *(Risk-sensitive Separation Principle). The evaluation of the extremized past stress and future stress can be decoupled if these calculations are made conditional on the current vector,* $\mathbf{z}_t$*. These partially extremized stress functions,* $P_t(\mathbf{z}_t, \mathbf{H}_t)$ *and* $F_t(\mathbf{z}_t)$*, relate to estimation and control respectively. In fact, the evaluation of* P_t *summarizes the effect of past observations, while the evaluation of* F_t *implies the calculation of the control* $\mathbf{u}_t(\mathbf{z}_t)$*, which would be optimal if* $\mathbf{z}_t$ *were known. The calculations of* $P_t(\mathbf{z}_t, \mathbf{H}_t)$ *and* $F_t(\mathbf{z}_t)$ *are then recoupled, maximizing* $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t)$ *with respect to* $\mathbf{z}_t$*; this yields the maximum total stress estimate (MTSE)* $\breve{\mathbf{z}}_t$*.*

- (Risk-sensitive Certainty Equivalence Principle). The optimal value of the control vector at time t *is then given by* $\mathbf{u}_t(\breve{\mathbf{z}}_t)$*.*

Thanks to the recursive structure of the Markovian LEQG problem, the extremized past and future stress respect the following recursions

$$P_t(\mathbf{z}_t, \mathbf{H}_t) = \max_{\mathbf{z}_{t-1}} \left[c_{t-1} - \frac{1}{\rho} d_t + P_{t-1}(\mathbf{z}_{t-1}, \mathbf{H}_{t-1}) \right], \tag{1.2}$$

$$F_t(\mathbf{z}_t) = \min_{\mathbf{u}_t} \left\{ \max_{\mathbf{z}_{t+1}, \mathbf{w}_{t+1}} \left[c_t - \frac{1}{\rho} d_{t+1} + F_{t+1}(\mathbf{z}_{t+1}) \right] \right\}, \tag{1.3}$$

with boundary conditions

$$P_0(\mathbf{z}_0, \mathbf{h}_0) = -\frac{1}{\rho} \boldsymbol{\mathcal{D}}_0 ,$$

$$F_T(\mathbf{z}_T) = \boldsymbol{\mathcal{C}}_T .$$

These past and future stress recursions are particularly useful in the evaluation of P_t and F_t and in the implementation of Theorem 2. Furthermore, as a corollary of this Theorem,

we note that under perfect state observation only the future stress recursion must be solved and therefore the calculation of the optimal control path is much easier. Considering the plant equation for the state vector, $\mathbf{z}_{t+1}$ can be substituted with $\boldsymbol{\epsilon}_{t+1}$ so that the future stress recursion is now as follows

$$F_t(\mathbf{z}_t) \quad = \quad \min_{\mathbf{u}_t} \left\{ \max_{\boldsymbol{\epsilon}_{t+1}} \left[c_t \, - \, \frac{1}{\rho} \, d_{t+1} \, + \, F_{t+1}(\mathbf{z}_{t+1}) \right] \right\}. \tag{1.4}$$

In conclusion, a recursion similar to the Bellman equation for the value function of dynamic programming is obtained: given the extremized future stress at time $t+1$, the optimal control at time t is obtained by solving the future stress recursion. Thus, first maximization is undertaken with respect to $\boldsymbol{\epsilon_{t+1}}$ and then the resulting value is minimized with respect to $\mathbf{u}_t$. Similarly to the Markovian LQG problem, the extremized future stress function is quadratic in the state vector, $F_t(\mathbf{z}_t) = \mathbf{z}_t' \boldsymbol{\Pi}_t \mathbf{z}_t$, while the optimal control rule is linear in the state vector, $\mathbf{u}_t = \mathbf{K}_t \mathbf{z}_t$, where $\boldsymbol{\Pi}_t$ and $\mathbf{K}_t$ respect recursions which correspond to modified versions of the *Riccati* recursions for the Markovian LQG problem. The following Theorem reveals the nexus with the common solutions to the standard Markovian LQG problem.

Theorem 3 - *(Risk-sensitive Riccati Equation). If the matrix* $\boldsymbol{\Pi}_{t+1} - (1/\rho)\, \mathbf{N}^{-1}$ *is negative definite, the solution to the extremization of the future stress exists and it is given by*

$$F_t(\mathbf{z}_t) \; = \; \mathbf{z}_t' \, \boldsymbol{\Pi}_t \, \mathbf{z}_t \quad \textit{and} \quad \mathbf{u}_t \; = \; \mathbf{K}_t \, \mathbf{z}_t \quad \textit{where}$$

$$\begin{aligned} \boldsymbol{\Pi}_t \quad &= \quad \mathbf{R} \, + \, \mathbf{A}' \widetilde{\boldsymbol{\Pi}}_{t+1} \mathbf{A} \, - \, (\mathbf{S}' \, + \, \mathbf{A}' \widetilde{\boldsymbol{\Pi}}_{t+1} \mathbf{B})(\mathbf{Q} \, + \, \mathbf{B}' \widetilde{\boldsymbol{\Pi}}_{t+1} \mathbf{B})^{-1} (\mathbf{S} \, + \, \mathbf{B}' \widetilde{\boldsymbol{\Pi}}_{t+1} \mathbf{A}) \, , \\ \mathbf{K}_t \quad &= \quad - \, (\mathbf{Q} \, + \, \mathbf{B}' \widetilde{\boldsymbol{\Pi}}_{t+1} \mathbf{B})^{-1} (\mathbf{S} \, + \, \mathbf{B}' \widetilde{\boldsymbol{\Pi}}_{t+1} \mathbf{A}) \quad \textit{and} \end{aligned}$$

$$\widetilde{\boldsymbol{\Pi}}_{t+1} \; = \; (\boldsymbol{\Pi}_{t+1}^{-1} \; - \; \rho \, \mathbf{N})^{-1} \, .$$

It is worth noticing that with respect to the standard Riccati equation which applies to the standard Markovian LQG problem with perfect state observation, the matrix $\boldsymbol{\Pi}_{t+1}$ is now replaced by the modified matrix $\widetilde{\boldsymbol{\Pi}}_{t+1}$. In other words, in the risk-sensitive case the optimal control retains a specification which is very similar to the one that would prevail in a risk-neutral environment (with $\rho = 0$), in that only a correction for the impact of uncertainty and risk-aversion must be inserted in the expressions for the recursions of $\boldsymbol{\Pi}_t$ and $\mathbf{K}_t$.[5]

[5]The requirement that the matrix $\widetilde{\boldsymbol{\Pi}}_{t+1}$ being negative definite derives from a second order condition which must hold for the total stress function to satisfy the saddle point condition imposed by Theorem 1. As noted

When the state vector is not perfectly observed the past stress recursion must also be solved. In this respect the following result holds.

Theorem 4 - *(Past Stress Maximization). Let* $\hat{\mathbf{z}}_0$ *and* $\mathbf{\Omega}_0$ *denote the mean vector and covariance matrix for* $\mathbf{z}_0$ *conditional on the initial information* $\mathbf{h}_0$*, then the extremized past stress is*

$$P_t(\mathbf{z}_t, \mathbf{H}_t) \;=\; -\,\frac{1}{\rho}\,(\mathbf{z}_t \,-\, \hat{\mathbf{z}}_t)'\,\mathbf{\Omega}_t^{-1}\,(\mathbf{z}_t \,-\, \hat{\mathbf{z}}_t) \;+\; \cdots\,,$$

where $+\cdots$ *denotes terms independent of the state vector, while* $\hat{\mathbf{z}}_t$ *denotes the maximum past stress estimate (MPSE) of the state vector at time* t*.*

- (Risk-sensitive Kalman Filter). The matrix $\mathbf{\Omega}_t$ *and the MPSE for the state vector* $\mathbf{z}_t$ *respect the following recursions,*

$$\begin{aligned}
\mathbf{\Omega}_t &= \mathbf{N} \,+\, \mathbf{A}\,\widetilde{\mathbf{\Omega}}_{t-1}\,\mathbf{A}' \,-\, (\mathbf{L} \,+\, \mathbf{A}\widetilde{\mathbf{\Omega}}_{t-1}\,\mathbf{C}')(\mathbf{M} \,+\, \mathbf{C}\,\widetilde{\mathbf{\Omega}}_{t-1}\,\mathbf{C}')^{-1}\,(\mathbf{L}' \,+\, \mathbf{C}\,\widetilde{\mathbf{\Omega}}_{t-1}\,\mathbf{A}')\,,\\
\hat{\mathbf{z}}_t &= \mathbf{A}\,\tilde{\mathbf{z}}_{t-1} \,+\, \mathbf{B}\,\mathbf{u}_{t-1} \,+\, (\mathbf{L} \,+\, \mathbf{A}\widetilde{\mathbf{\Omega}}_{t-1}\,\mathbf{C}')(\mathbf{M} \,+\, \mathbf{C}\,\widetilde{\mathbf{\Omega}}_{t-1}\,\mathbf{C}')^{-1}\,(\mathbf{w}_t \,-\, \tilde{\mathbf{z}}_{t-1})\,,
\end{aligned}$$

where, under the condition that $\mathbf{\Omega}_{t-1}^{-1} - \rho\mathbf{R}$ *is positive definite,*

$$\begin{aligned}
\widetilde{\mathbf{\Omega}}_{t-1} &= (\mathbf{\Omega}_{t-1}^{-1} \,-\, \rho\,\mathbf{R})^{-1}\,,\\
\tilde{\mathbf{z}}_{t-1} &= (\mathbf{\Omega}_{t-1}^{-1} \,-\, \rho\,\mathbf{R})^{-1}\,(\mathbf{\Omega}_{t-1}^{-1}\,\hat{\mathbf{z}}_{t-1} \,+\, \rho\,\mathbf{S}'\,\mathbf{u}_{t-1})\,.
\end{aligned}$$

Theorem 4 reinforces the nexus with the risk-neutral case. In fact, the MPSE for the Markovian LEQG problem respects a recursion which is similar to that obtained applying the Kalman filter in the standard Markovian LQG problem. Indeed, with respect to the Kalman filter which allows to derive the MLE for the state vector (and the corresponding Riccati equation for the matrix $\mathbf{\Omega}_t$) a straightforward correction must be introduced to accommodate the impact of risk-aversion. This is achieved by simply replacing $\mathbf{z}_{t-1}$ and $\mathbf{\Omega}_{t-1}$ in the standard Kalman filter and Riccati equation of the risk-neutral case with respectively $\tilde{\mathbf{z}}_{t-1}$ and $\widetilde{\mathbf{\Omega}}_{t-1}$. It is however important to notice that $\hat{\mathbf{z}}_t$ is not longer a conditional expectation and consequently $\mathbf{\Omega}_t$ is not longer a covariance matrix in the standard sense which applies under risk-neutrality.

To re-couple the extremization of past and future stress, we apply Theorem 2, so that the current value of the state vector is replaced with its MPSE in $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t)$. The resulting function is then maximized with respect to state vector $\mathbf{z}_t$ to obtain the maximum total stress

by Whittle, whenever the cost function c_t is non-negative such condition fails for ρ large enough.

estimate (MTSE), $\breve{\mathbf{z}}_t$. This is given by the following expression

$$\breve{\mathbf{z}}_t \;=\; (I \,-\, \rho\, \mathbf{\Omega}_t\, \mathbf{\Pi}_t)^{-1}\, \hat{\mathbf{z}}_t.$$

Finally, the optimal control in the imperfect state observation case is given by Theorem 3 where $\breve{\mathbf{z}}_t$ replaces $\mathbf{z}_t$.

We now propose a couple of specific examples in finance and economics. This allows to adapt the linear exponential quadratic Gaussian problem to the specificities of economic investigation. We start from a classic example in finance, that of the optimal trading strategy on the part of a risk-averse agent who possesses private information on the liquidation value of a group of risky assets. We investigate her behavior within the trading protocol of the sequential auction market described by Kyle (Kyle, 1985).

2 Risk-averse Insider Trading in Sequential Auction Markets

In this Section we concentrate on an extension of a very important contribution to financial economics due to Kyle (1985). The main innovation in his model is that of providing a very useful analytical framework which has been widely used in the recent market micro structure literature to analyze the link between market organization, strategic behavior and the informational role of prices in securities markets. This analytical framework is particularly elegant and powerful. It is elegant as simple to interpret equilibria are obtained. Furthermore, a series of characteristics of securities markets, such as their efficiency and liquidity, are endogenously defined. It is also powerful because it preserves the linearity of the equilibria.

Kyle investigates the behavior of an informed trader (insider) in a securities market. Such an agent presents an incentive to act strategically and exploit her informational advantage to gain speculative profits from her trading activity. She acts strategically because in choosing the timing and size of her transactions she takes into account the impact that her trades will have on the equilibrium price and hence on her profits. In studying the optimal behavior of the insider, Kyle assumes such an agent is risk-neutral. Subrahmanyam (1991) and Holden and Subrahmanyam (1994) have instead considered the scenario with risk-averse insiders. We recast their analysis within the risk-sensitive optimal control framework, extending it to a multi-security version of Kyle's model as in Caballè and Krishnan (1994).[6]

[6]Vitale (Vitale, 1995) applies Whittle's methodology to Kyle's model with a risky asset and a single risk-

2.1 Strategic Informed Trading

In a securities market a market maker trades with a group of customers a risk-free bond (or numeraire), which pays a certain rate of return (normalized to 0), for a group of m risky assets with uncertain liquidation values. Clients include an insider, who knows the liquidation value of the m risky assets, and a group of liquidity (noise) traders, who trade purely for liquidity reasons. The liquidation vector $\tilde{\boldsymbol{v}}$, i.e. the vector of liquidation values of the m risky assets, is determined at time 0, before trading in the market starts, and is publicly announced at time 1, when no more trading is possible. Apart from the insider, no one knows the actual realization of $\tilde{\boldsymbol{v}}$ at time 0. However, unconditionally $\tilde{\boldsymbol{v}} \sim \boldsymbol{\mathcal{N}}(\boldsymbol{\mu}_v, \boldsymbol{\Sigma})$. This information is common knowledge.

Between the instant the liquidation vector is realized and that in which it is announced N rounds of trading, in the form of call auctions, are conducted by the market maker. Any call auction is identified by the subscript n and takes place at time t_n, with $0 < t_1 < \ldots < t_N < 1$. When auction n is called at time t_n, the market maker's clients, the liquidity traders and the insider, select their market orders, that is the amount of the m risky assets they desire to purchase or sell. We indicate with the vectors $\Delta\tilde{\boldsymbol{x}}_n^l$ and $\Delta\tilde{\boldsymbol{x}}_n^i$ the market orders of respectively the liquidity traders and the insider.[7] These orders are batched together and the overall market order, $\Delta\tilde{\boldsymbol{x}}_n = \Delta\tilde{\boldsymbol{x}}_n^i + \Delta\tilde{\boldsymbol{x}}_n^l$, is passed to the market maker. He then fixes a vector of transaction prices, $\tilde{\boldsymbol{p}}_n$, at which, using his own inventories, he executes all the individual orders.

Notice that the market maker cannot observe either the individual orders or the identity of his clients. This will permit the insider to exploit over time her informational advantage. The market maker is risk-neutral. Bertrand competition in the market making industry will force him to set the transaction prices, relative to the numeraire, for the m risky assets according to a semi-strong form efficiency condition. Since he just observes the flow of total market orders he receives along the sequence of auctions and since these market orders may contain some information, due to the presence of the insider, we can state that at any round of trading, n, the vector of transaction prices, $\tilde{\boldsymbol{p}}_n$, is

$$\tilde{\boldsymbol{p}}_n \;=\; E[\tilde{\boldsymbol{v}} \mid \Delta\tilde{\boldsymbol{x}}_1, \ldots, \Delta\tilde{\boldsymbol{x}}_{n-1}, \Delta\tilde{\boldsymbol{x}}_n]\,. \tag{2.1}$$

averse insider.

[7] A market order is an order to purchase or sell a given quantity of an asset. Such an order does not indicate an explicit price at which the transaction must be executed. However, it must be executed at the best available ask or bid price. By convention when $\Delta\tilde{x}_j^s$ (with $1 \leq j \leq m$ and $s = i, l$) is positive (negative) agent s purchases (sells) risky security j.

The liquidity traders are supposed to place unpredictable market orders. Thus, in any round n, their vector of market orders is supposed to be distributed as a multi-normal, $\Delta\tilde{\boldsymbol{x}}_n^l \sim \boldsymbol{\mathcal{N}}(\mathbf{0}, \boldsymbol{\Sigma}_l \Delta_n)$, where $\Delta_n = t_n - t_{n-1}$ and the covariance matrix $\boldsymbol{\Sigma}_l$ is diagonal. The random vectors $\{\Delta\tilde{\boldsymbol{x}}_n^l\}_{n=1}^N$ are independent among each other and of the liquidation vector $\tilde{\boldsymbol{v}}$. Because of these assumptions the market orders of the liquidity traders follow a multi-dimensional white noise process.[8]

Differently from Kyle, we assume the insider is risk-averse. Since she does not have any initial endowment of the risky assets, she maximizes the expected utility she will receive from the final value of her trading profits. Since the prices charged by the market maker are function of the unpredictable orders of the liquidity traders, the profits of the insider are not certain. They are given by the following expression $\tilde{\pi}^i = \sum_{n=1}^N (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{p}}_n)' \Delta\tilde{\boldsymbol{x}}_n^i$. Assuming she is endowed with a CARA utility function with coefficient ρ, when auction n is called she solves the following *program*

$$\Delta\tilde{\boldsymbol{x}}_n^i \;=\; \text{argmax}\; E[-\exp(-\rho\,\tilde{\pi}^i) \mid \tilde{\boldsymbol{p}}_1, \tilde{\boldsymbol{p}}_2, \ldots, \tilde{\boldsymbol{p}}_{n-1}, \tilde{\boldsymbol{v}}]\,. \tag{2.2}$$

Consider that the insider needs to solve an optimal control problem characterized by a clear trade-off. In fact, a larger market order today generates larger profits now at the expense of future ones, since a more informative order is passed to the market maker reducing his uncertainty on the liquidation vector. On the other hand, the market maker needs to solve a filtering problem. He uses the signal contained in the flow of orders to up-date his expectation of the liquidation vector. This will induce a process of convergence of the vector of transaction prices to the vector of actual liquidation values.

To solve simultaneously and consistently these two problems Kyle introduces a special notion of sequential equilibrium. We adapt it to the scenario in which the insider is risk-averse. First, we need to define the strategies that characterize an equilibrium. These are two collections of functions, $\mathbf{X}$ and $\mathbf{P}$, that indicate the trading strategy of the insider and the pricing rule of the market maker for any round of trading n,

$$\mathbf{X} \;=\; \langle \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n, \ldots, \mathbf{X}_N \rangle\,, \qquad \mathbf{P} \;=\; \langle \mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n, \ldots, \mathbf{P}_N \rangle\,, \quad \text{where} \tag{2.3}$$

$$\Delta\tilde{\boldsymbol{x}}_n^i \;=\; \mathbf{X}_n(\tilde{\boldsymbol{p}}_1, \ldots, \tilde{p}_{n-1}, \tilde{\boldsymbol{v}}), \qquad \tilde{\boldsymbol{p}}_n \;=\; \mathbf{P}_n(\Delta\tilde{\boldsymbol{x}}_1, \ldots, \Delta\tilde{\boldsymbol{x}}_n)\,. \tag{2.4}$$

The insider's profits, $\tilde{\pi}^i$, is function of these two strategies, $\tilde{\pi}^i = \tilde{\Pi}^i(\mathbf{X}, \mathbf{P})$. We can now define

[8] Our analytical solution also works when the covariance matrix $\boldsymbol{\Sigma}_l$ is not diagonal. However, given the nature of liquidity trading it is more reasonable to assume that it is segmented across the m risky assets.

a sequential auction equilibrium.

Definition 4 *A sequential auction equilibrium is a couple* $(\mathbf{X}, \mathbf{P})$ *such that the following two conditions hold:*
(1) For all $n = 1, \ldots, N$ *the insider maximizes her expected utility as in the program (2.2). That is,* $\forall n = 1, \ldots, N$ *and for any alternative trading strategy* $\mathbf{X}'$ *such that* $\mathbf{X}'_1 = \mathbf{X}_1$, $\mathbf{X}'_2 = \mathbf{X}_2, \ldots,$ $\mathbf{X}'_{n-1} = \mathbf{X}_{n-1}$,

$$E[-\exp(-\rho\,\tilde{\Pi}^i(\mathbf{X}, \mathbf{P})) \mid \tilde{\boldsymbol{p}}_1, \ldots, \tilde{\boldsymbol{p}}_{n-1}, \tilde{\boldsymbol{v}}\,] \;\geq\; E[-\exp(-\rho\,\tilde{\Pi}^i(\mathbf{X}', \mathbf{P})) \mid \tilde{\boldsymbol{p}}_1, \ldots, \tilde{\boldsymbol{p}}_{n-1}, \tilde{\boldsymbol{v}}\,]\,. \quad (2.5)$$

(2) For all $n = 1, \ldots, N$, *the market maker sets the vector of transaction prices according to the efficiency condition (2.1). That is* $\forall n = 1, \ldots, N$

$$\tilde{\boldsymbol{p}}_n \;=\; E[\tilde{\boldsymbol{v}} \mid \Delta\tilde{\boldsymbol{x}}_1, \ldots, \Delta\tilde{\boldsymbol{x}}_{n-1}, \Delta\tilde{\boldsymbol{x}}_n]\,. \quad (2.6)$$

We can then define a Markovian linear equilibrium as follows.[9]

Definition 5 *A sequential auction equilibrium is linear if the component functions of strategies* $\mathbf{X}$ *and* $\mathbf{P}$ *are linear. A linear sequential auction equilibrium is Markovian if there exist matrices of constants* $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Lambda}_2, \ldots,$ $\boldsymbol{\Lambda}_N$, *such that for any* $n = 1, \ldots, N$

$$\tilde{\boldsymbol{p}}_n \;=\; \tilde{\boldsymbol{p}}_{n-1} \;+\; \boldsymbol{\Lambda}_n\,\Delta\tilde{\boldsymbol{x}}_n\,. \quad (2.7)$$

2.2 A Markovian Linear Equilibrium

To find a Markovian linear equilibrium for the multi-asset version of Kyle's model with a risk-averse insider, let us concentrate on the insider's trading strategy. Let us assume the market maker sets the vector of transaction prices according to equation (2.7). Then, we can easily recast the optimization exercise of the insider within the LEGQ framework. In fact, let us define the state vector $\mathbf{z}_n \equiv \boldsymbol{p}_{n-1} - \boldsymbol{v}$, the control variable $\mathbf{u}_n \equiv \Delta\boldsymbol{x}^i_n$ and the noise vector $\boldsymbol{\epsilon}_n \equiv \boldsymbol{\Lambda}_{n-1}\,\Delta\boldsymbol{x}^l_{n-1}$, for $n = 1, \ldots, N$. Then, under equation (2.7) we can write the plant equation as follows

$$\mathbf{z}_{n+1} \;=\; \mathbf{z}_n \;+\; \boldsymbol{\Lambda}_n\,\mathbf{u}_n \;+\; \boldsymbol{\epsilon}_{n+1}\,. \quad (2.8)$$

[9] For completeness we should specify the beliefs of the market maker and insider. They are based on the true unconditional distributions and on the application of Bayes' theorem. Furthermore, we do not need to define out-of-equilibrium beliefs for our agents. In fact, because of the normality assumption, there cannot be out-of-equilibrium observations.

In addition, we can define the cost function $c_n \equiv 2(\mathbf{z}_n + \mathbf{\Lambda}_n\,\mathbf{u}_n + \boldsymbol{\epsilon}_{n+1})'\mathbf{u}_n$. It is immediate to notice that $\pi^i = -(1/2)\sum_{n=1}^N c_n$. This implies that we can solve the insider optimization problem by minimizing the criterion (1.1) under the plant equation (2.8) for $\boldsymbol{\mathcal{C}} = \sum_{n=1}^N c_n$. In other words, the insider needs solving a LEQG optimal control problem, even if this is not in its standard Markovian format, in that the cost function component c_n is not just a quadratic form of current state and control vectors, $\mathbf{z}_n$ and $\mathbf{u}_n$. The extra term $\boldsymbol{\epsilon}'_{n+1}\mathbf{u}_n$ means that we cannot exploit the recursive risk-sensitive equations in Theorem 3 to obtain automatically the vector of optimal market orders of the insider at call auction n. However, we can apply the risk-sensitive SP in Theorem 2. Furthermore, noticing that there is perfect state observation on the part of the insider, we need solving only the future stress recursion,

$$F_n(\mathbf{z}_n) \;=\; \min_{\mathbf{u}_n}\left\{\max_{\boldsymbol{\epsilon}_{n+1}}\left[c_n \;-\; \frac{1}{\rho}\,d_{n+1} \;+\; F_{n+1}(\mathbf{z}_{n+1})\right]\right\}\,,$$

where $d_{n+1} = \boldsymbol{\epsilon}'_{n+1}\mathbf{N}_{n+1}^{-1}\boldsymbol{\epsilon}_{n+1}$ and $\mathbf{N}_{n+1} \equiv \mathbf{\Lambda}_n\mathbf{\Sigma}_l\Delta_n\mathbf{\Lambda}'_n$. We can assume that the extremized future stress function is quadratic in the state vector, so that $F_{n+1}(\mathbf{z}_{n+1}) \equiv \mathbf{z}'_{n+1}\mathbf{\Pi}_{n+1}\mathbf{z}_{n+1}$, where $\mathbf{\Pi}_{n+1}$ is a negative definite matrix and $\mathbf{\Pi}_{N+1} = \mathbf{0}$.

Solving the maximization problem with respect to $\boldsymbol{\epsilon}_{n+1}$ in the recursion it is found that, for $\widetilde{\mathbf{\Pi}}_{n+1} = \mathbf{\Pi}_{n+1}^{-1} - \rho\,\mathbf{N}_{n+1}$ negative definite and assuming that $\mathbf{\Lambda}_n$ is symmetric, a maximum is reached for[10]

$$\hat{\boldsymbol{\epsilon}}_{n+1} \;=\; \rho\,\mathbf{N}_{n+1}\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\left(\mathbf{z}_n \;+\; \widetilde{\mathbf{\Lambda}}_n\,\mathbf{u}_n\right)\,,$$

where $\widetilde{\mathbf{\Lambda}}_n = \mathbf{\Lambda}_n + \mathbf{\Pi}_{n+1}^{-1}$. Inserting $\hat{\boldsymbol{\epsilon}}_{n+1}$ into the future stress recursion and minimizing with respect to $\mathbf{u}_n$ we find, for $-\,\mathbf{\Pi}_{n+1}^{-1} + \widetilde{\mathbf{\Lambda}}_n\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,\widetilde{\mathbf{\Lambda}}'_n$ positive definite, a minimum for

$$\hat{\mathbf{u}}_n \;=\; -\,\boldsymbol{\mathcal{B}}_n\,\mathbf{z}_n\,, \qquad \text{where} \quad \boldsymbol{\mathcal{B}}_n \;=\; \left(-\,\mathbf{\Pi}_{n+1}^{-1} \;+\; \widetilde{\mathbf{\Lambda}}_n\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,\widetilde{\mathbf{\Lambda}}'_n\right)^{-1}\widetilde{\mathbf{\Lambda}}_n\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,. \tag{2.9}$$

Inserting $\hat{\mathbf{u}}_n$ into the future stress recursion we find that $F_n(\mathbf{z}_n) = \mathbf{z}'_n\mathbf{\Pi}_n\mathbf{z}_n$, where

$$\mathbf{\Pi}_n \quad = \quad \widetilde{\mathbf{\Pi}}_{n+1}^{-1} \;-\; \widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,\widetilde{\mathbf{\Lambda}}'_n\left(-\,\mathbf{\Pi}_{n+1}^{-1} \;+\; \widetilde{\mathbf{\Lambda}}_n\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,\widetilde{\mathbf{\Lambda}}'_n\right)^{-1}\widetilde{\mathbf{\Lambda}}_n\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,. \tag{2.10}$$

Notice that this solution implies that

$$\Delta\tilde{\boldsymbol{x}}_n^i \quad = \quad \boldsymbol{\mathcal{B}}_n\,(\tilde{\boldsymbol{v}} \;-\; \tilde{\boldsymbol{p}}_{n-1})\,, \tag{2.11}$$

so that the optimal trading strategy of the insider entails that her market orders are linear in

[10] In the Appendix we spell out these and other calculations.

the subjective mis-pricing of the risky assets, $\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{p}}_{n-1}$. Interestingly, this extends to the multi-asset formulation a result Kyle originally establishes under risk-neutrality for the formulation with only one risky asset. Turning to the filtering problem of the market maker, assuming that the market orders of the insider are given by this linear rule, by applying the projection Theorem for normally distributed random variables, we find that

$$\tilde{\boldsymbol{p}}_n \quad = \quad \tilde{\boldsymbol{p}}_{n-1} \; + \; \boldsymbol{\Lambda}_n \, \Delta \tilde{\boldsymbol{x}}_n \, , \tag{2.12}$$

where

$$\boldsymbol{\Lambda}_n \quad = \quad \boldsymbol{\Sigma}_{n-1} \, \boldsymbol{\mathcal{B}}_n' \left(\boldsymbol{\mathcal{B}}_n \, \boldsymbol{\Sigma}_{n-1} \, \boldsymbol{\mathcal{B}}_n' \; + \; \boldsymbol{\Sigma}_l \Delta_n \right)^{-1} \tag{2.13}$$

and $\tilde{\boldsymbol{p}}_0 = \boldsymbol{\mu}_v$. Here $\tilde{\boldsymbol{p}}_n$ corresponds to the vector of expected liquidation values of the m risky assets given the information the market maker possesses at the end of round of trading n, while $\boldsymbol{\Sigma}_n$ is the corresponding conditional covariance matrix. From the projection Theorem we know that this is equal to

$$\boldsymbol{\Sigma}_n \quad = \quad \boldsymbol{\Sigma}_{n-1} \; - \; \boldsymbol{\Sigma}_{n-1} \, \boldsymbol{\mathcal{B}}_n' \left(\boldsymbol{\mathcal{B}}_n \, \boldsymbol{\Sigma}_{n-1} \, \boldsymbol{\mathcal{B}}_n' \; + \; \boldsymbol{\Sigma}_l \Delta_n \right)^{-1} \boldsymbol{\mathcal{B}}_n \, \boldsymbol{\Sigma}_{n-1} \, . \tag{2.14}$$

2.3 The Equilibrium Properties

The condition that $-\boldsymbol{\Pi}_{n+1}^{-1} + \widetilde{\boldsymbol{\Lambda}}_n \, \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1} \, \widetilde{\boldsymbol{\Lambda}}_n'$ being positive definite guarantees that the expected profits of the insider are bounded and that a maximum for her expected utility exists. Intuitively, such condition rules out situations in which the informed trader destabilizes the asset prices in the initial call auctions with large unprofitable market orders and then gains huge benefits in the following periods. In fact, when $\boldsymbol{\Lambda}_n$ is "large" and $\boldsymbol{\Pi}_{n+1}$ is "small", small orders are sufficient and relatively "inexpensive" in terms of forgone utility to destabilize the asset prices in auction n. In this case the destabilization can take place. Instead, if $\boldsymbol{\Pi}_{n+1}$ is "large" with respect to $\boldsymbol{\Lambda}_n$ it is not convenient for the insider to move the asset prices away from their liquidation value, because the cost-opportunity of doing so, measured by $\boldsymbol{\Pi}_{n+1}$, is too large. The second order condition simply places an upper limit to the admissible value of $\boldsymbol{\Lambda}_n$ ruling out these destabilizing schemes.

In characterizing the equilibrium we impose the condition that $\boldsymbol{\Lambda}_n$ is symmetric. This symmetry can be established. Consider in fact the last round of trading N. We see immediately

that $\boldsymbol{\mathcal{B}}_N = (2\boldsymbol{\Lambda}_N + \rho\mathbf{N}_{N+1})^{-1}$, from which we conclude that

$$\boldsymbol{\Lambda}_N \quad = \quad \boldsymbol{\Sigma}_{N-1}\left(2\boldsymbol{\Lambda}_N + \rho\mathbf{N}_{N+1}\right)^{'-1}\left((2\boldsymbol{\Lambda}_N + \rho\mathbf{N}_{N+1})^{-1}\boldsymbol{\Sigma}_{N-1}(2\boldsymbol{\Lambda}_N + \rho\mathbf{N}_{N+1})^{'-1} + \boldsymbol{\Sigma}_l\Delta_N\right)^{-1}.$$

With some manipulations we can write this as follows

$$\boldsymbol{\mathcal{M}}'_N\,\boldsymbol{\Lambda}'_N\,\boldsymbol{\Sigma}_{N-1}\,\boldsymbol{\Lambda}_N\,\boldsymbol{\mathcal{M}}_N \quad = \quad \boldsymbol{\Sigma}_l^{-1}\Delta_N^{-1}\left(\boldsymbol{\mathcal{M}}_N - \mathbf{I}_m\right),$$

where $\boldsymbol{\mathcal{M}}_N \equiv 2\mathbf{I}_m + \rho\boldsymbol{\Sigma}_l\Delta_N\boldsymbol{\Lambda}'_N$. As the left hand side is symmetric, so is the right hand side. This can be written as $\boldsymbol{\Sigma}_l^{-1}\Delta_N^{-1} + \rho\boldsymbol{\Lambda}'_N$, so that $\boldsymbol{\Lambda}_N$ must be symmetric. A similar argument holds for $n < N$. This symmetry implies that whatever the characteristics of the m assets, the sensitiveness (or liquidity coefficient), $\lambda_n^{j,h}$, of the price of asset j, $\tilde{p}_{j,n}$, to asset h-order flow, $\Delta\tilde{x}_{h,n}$, is always equal to the sensitiveness, $\lambda_n^{h,j}$, of the price of asset h, $\tilde{p}_{h,n}$, to asset j-order flow, $\Delta\tilde{x}_{j,n}$.

In a static risk-neutral set-up Caballè and Krishnan (Caballè and Krishnan, 1994) reach the same result and suggest that this shows how in equilibrium the insider finds it optimal to make all market orders equivalently informative.[11] In other words, if the liquidity traders are more active in the market for asset j then that for asset h, the insider can better camouflage her trading in the former market and consequently will trade more aggressively. In equilibrium, she will trade in such a way that the information content of the two order flows is the same. This means that order flow $\Delta\tilde{x}_{j,n}$ is as useful in predicting the liquidation value $\tilde{v}_h$ as order flow $\Delta\tilde{x}_{h,n}$ in predicting the liquidation value $\tilde{v}_j$. Given semi-strong efficiency condition transaction prices correspond to regression functions and hence the matrix $\boldsymbol{\Lambda}_n$ of the liquidity coefficients is equivalent to the ratio between prior-to-posterior precisions. Then, as the prior covariance matrix, $\boldsymbol{\Sigma}_{N-1}$, is symmetric, given the same degree of improvement in the precisions from the individual order flows, this symmetry is preserved in the pricing rule.

For $m = 1$, it is easier to interpret the market equilibrium described in equations (2.11) to (2.14). In this case, the state vector collapses to the scalar $z_n = p_{n-1} - v$ so that $\boldsymbol{\Pi}_n$ corresponds to a coefficient that we write as $-2\alpha_{n-1}$. This means that $F_n(z_n) = -2\alpha_{n-1}(p_{n-1} - v)^2$. Then, from equations (2.9), (2.10), (2.13) and (2.14), we conclude that $\Delta\tilde{x}_n^i = \beta_n(\tilde{v} - \tilde{p}_{n-1})$ and

[11] It can be shown that for $\rho = 0$ $\boldsymbol{\Lambda}_N$ collapses to the expression derived by Caballè and Krishnan for the scenario with one insider, $\boldsymbol{\Lambda}_N = (1/2)(\boldsymbol{\Sigma}_l\Delta_N)^{-1/2}((\boldsymbol{\Sigma}_l\Delta_N)^{1/2}\boldsymbol{\Sigma}_{N-1}(\boldsymbol{\Sigma}_l\Delta_N)^{1/2})^{1/2}(\boldsymbol{\Sigma}_l\Delta_N)^{-1/2}$.

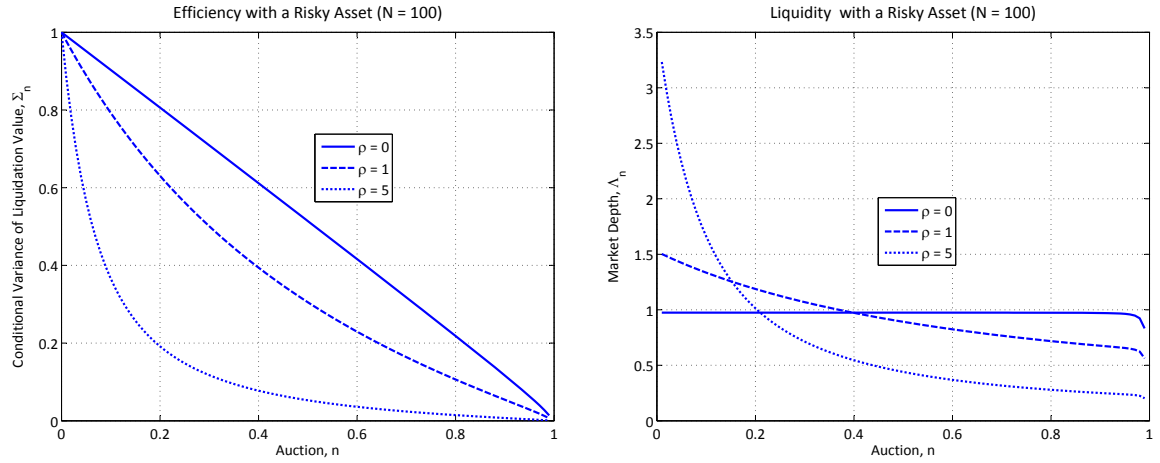

Figure 1: The Dynamics of efficiency and liquidity for $m = 1$, $\sigma_l^2 = \Sigma = 1$.

$\tilde{p}_n = \tilde{p}_{n-1} + \lambda_n \Delta\tilde{x}_n$, where

$$\begin{aligned}
\beta_n &= \frac{1 - 2\alpha_n\lambda_n}{2\lambda_n(1 - \alpha_n\lambda_n) + \lambda_n^2\rho\sigma_l^2\Delta_n}\,, \\
\alpha_{n-1} &= \frac{1}{2[2\lambda_n(1 - \alpha_n\lambda_n) + \lambda_n^2\rho\sigma_l^2\Delta_n]}\,, \\
\lambda_n &= \frac{\beta_n\Sigma_{n-1}}{\beta_n^2\Sigma_{n-1} + \sigma_l^2\Delta_n}\,, \\
\Sigma_n &= (1 - \beta_n\lambda_n)\Sigma_{n-1}\,.
\end{aligned}$$

Reassuringly, this is equivalent to the solution proposed by Holden and Subrahmanyam (1994) for the scenario with one insider. Moreover, it subsumes that proposed by Kyle for $\rho = 0$. From the expression for β_n we see that risk-aversion makes the insider care about the variance of her profits. The uncertainty the insider faces results from the randomness of the liquidity traders' orders. Given her information set at the onset of call auction n, the expected price is $E[\tilde{p}_n \mid \mathcal{I}_n^i] = \tilde{p}_{n-1} + \lambda_n\Delta x_n^i$ and consequently the conditional variance of this value is $\mathrm{Var}\,[\tilde{p}_n \mid \mathcal{I}_n^i] = \lambda_n^2\sigma_l^2\Delta_n$. This conditional variance and the insider's risk-aversion enter into the specification of the coefficient β_n, which defines the optimal trading rule of the insider, and hence affect the sequential auction equilibrium.

The expression for Σ_n indicates that the conditional variance of the liquidation value given the information set of the market maker is monotonically decreasing with n. This shows that

information is gradually incorporated into the asset price as it is disclosed through time by order flow. How quickly the transaction price converges to the true liquidation value depends on the insider's trading strategy. For $\rho = 0$, i.e. in Kyle's original formulation, information is disclosed at a constant speed (the derivative of Σ_n with respect to n is constant). This is because the insider finds it optimal to trade with constant intensity and maintain overtime a stable news-to-noise ratio in order flow. Consequently the price sensitiveness, or liquidity coefficient, λ_n is constant throughout most of the auctions.

Figure 1 shows that in the risk-averse case, instead, the informed trader places larger market orders in the initial auctions, as β_n is larger than for $\rho = 0$ and so is the liquidity coefficient, λ_n.[12] This is because the inter-temporal substitution between present and future profits is reduced by risk-aversion and, therefore, the insider prefers exploiting her information advantage earlier. Consequently, for $\rho > 0$ she trades more aggressively, order flow is more informative and the market maker learns at a higher speed the liquidation value of the asset. This implies that the conditional variance, Σ_n, declines more rapidly.

As the market maker progressively learns the liquidation value, price volatility and the uncertainty over future profits fall and consequently the inter-temporal substitution between present and future profits dissipates. Hence, as the last call auction approaches the impact of risk-aversion on the trading activity of the insider resembles that of the static version of Kyle's model studied by Subrahmanyam (1991): risk-aversion induces the insider to be more cautious and trade less aggressively. This means that as time elapses the informational content of order flow decreases (λ_n declines through time) and hence the reduction in the value of Σ_n is smaller. In the end, in the risk-averse case the information gain from order flow becomes smaller than that of the risk-neutral one while market liquidity is larger (λ_n is now smaller for $\rho > 0$). Anyway, despite the reduction in the information gain, the informativeness of prices is always larger in the risk-averse case as Σ_n is always smaller for ρ larger than 0.

Similar results are obtained with $m > 1$. In particular, in Figure 2 we consider a formulation with two risky assets whose liquidation values are positively correlated (the correlation coeffi-

[12]The algorithm we implement to find the solution of the non-linear difference equations for the parameters of the equilibrium is simple. Given α_n and Σ_n, there is a unique positive value of λ_n satisfying the condition $\lambda_n(1-\alpha_n\lambda_n)+\frac{1}{2}\rho\sigma_l^2\Delta_n\lambda_n^2 > 0$ that the optimization problem of the insider must satisfy. This value is given by the appropriate root of the the following equation $2(1-\alpha_n\lambda_n)(\Sigma_n-\sigma_l^2\Delta_n\lambda_n^2) = \Sigma_n + \rho\sigma_l^4\Delta_n^2\lambda_n^3$, which is obtained by substituting out the expression for β_n into λ_n. It is then immediate to obtain β_n and, through backward iteration, Σ_{n-1} and α_{n-1}. Since we have the final value $\alpha_N = 0$, we can define a numerical function of Σ_N, G, that gives the initial variance of the liquidation value $\Sigma' = G(\Sigma_N)$. Given that $G(\Sigma_N)$ is increasing in Σ_N it is easy to find via the Newton-Raphson method the root of the numerical equation $\Sigma = G(\Sigma_N)$ that gives the unique value of Σ_N consistent with the boundary value Σ. A similar procedure is followed for $m > 1$.

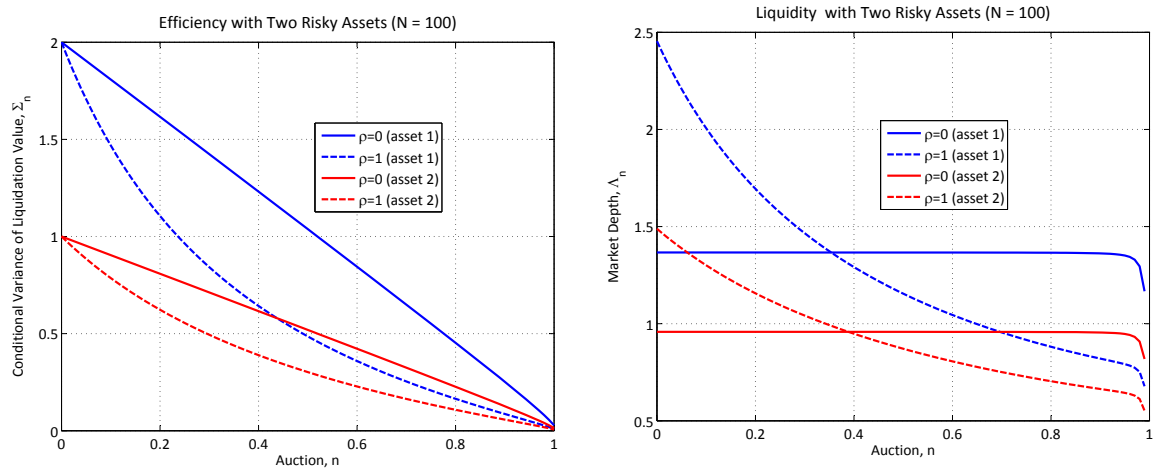

Figure 2: The Dynamics of efficiency and liquidity for $m = 2$, $\mathbf{\Sigma}_l = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{\Sigma} = \begin{pmatrix} 2 & 0.3\sqrt{2} \\ 0.3\sqrt{2} & 1 \end{pmatrix}$.

cient is equal to 0.3). In the left panel we plot the conditional variance of the two risky assets, $\Sigma_n^{1,1}$ and $\Sigma_n^{2,2}$. These values correspond to the elements in main diagonal of the covariance matrix $\mathbf{\Sigma}_n$. In the right panel we plot the corresponding liquidity coefficients $\lambda_n^{1,1}$ and $\lambda_n^{2,2}$, given by the elements in main diagonal of the matrix $\mathbf{\Lambda}_n$. The most notable conclusion that we draw from the two plots is that efficiency and liquidity conditions in the two markets resemble those observed in the scalar case. Thus, even in the multi-asset formulation a risk-neutral insider consumes her informational advantage at a constant pace, while a risk-averse one consumes her informational advantage at a faster pace to reduce the uncertainty over her future profits. As a consequence, while liquidity conditions are stable with a risk-neutral insider, when she is risk-averse liquidity conditions improve overtime as the pace at which her private information is revealed diminishes. In addition, transaction costs are larger in the market where the informational advantage of the insider is more pronounced, in that in such a market adverse selection is more severe.

A second important result pertains to the information spill-over among the two markets. As the liquidation values of the two risky-assets are correlated an informative signal on asset 1 (2) is also informative on asset 2 (1). The insider takes into account this information spill-over,

in that for $m = 2$ she chooses her market orders according to the following specification

$$\begin{aligned}\tilde{x}^i_{1,n} &= \beta^{1,1}_n(\tilde{v}_1 - p_{1,n-1}) + \beta^{1,2}_n(\tilde{v}_2 - p_{2,n-1}),\\ \tilde{x}^i_{2,n} &= \beta^{2,1}_n(\tilde{v}_1 - p_{1,n-1}) + \beta^{2,2}_n(\tilde{v}_2 - p_{2,n-1}).\end{aligned}$$

For a positive correlation between the two assets, the insider chooses negative values for the coefficients $\beta^{1,2}_n$ and $\beta^{2,1}_n$ reducing the information spill-over between the two markets.

In this respect it is useful to compare the properties of the equilibrium with those which prevail when the the liquidation values of the two assets are independent. In this case, there is perfect segmentation between the two markets and the insider can choose her optimal trading strategy for asset 1 irrespective of what takes place in the market for asset 2 and *viceversa*. Numerical analysis shows that the two markets turn out to be slightly more efficient when the two liquidation values are correlated than when the two markets are perfectly segmented. This suggests that the insider is capable of reducing the information spill-over between the two markets to the minimum.

3 LEQG Problems with Time-Discounting

Economic agents typically maximize time-separable utility functions with time-discounting. Then, one might want to see how discounting can be introduced properly in LEQG optimal control problems. Hansen and Sargent (Hansen and Sargent, 1994, 2005) have shown how discounted LEQG problems can be formulated and how recursive linear optimal control rules can be derived. In this Section we see how the recursive criterion *à la* Epstein Zin proposed by Hansen and Sargent can be modified in such a way that Whittle's recursive methodology can be salvaged nearly unchanged to accommodate time-discounting.

3.1 A Recursive Minimization Criterion Under Perfect State Observation

Consider an agent minimizing the following recursive criterion over the periods $t = 1, 2, \ldots, T$

$$\boldsymbol{\mathcal{V}}_t = \min_{\mathbf{u}_t}\left\{\frac{\rho}{2}c_t + \ln\left(E_t\left[\exp\left(\delta\,\frac{\rho}{2}\,\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right\}. \tag{3.1}$$

The coefficient δ is the discount factor such that $0 < \delta < 1$, the cost function, c_t, is quadratic in the control vector and the state vector, $c_t = \mathbf{u}_t'\mathbf{Q}\mathbf{u}_t + \mathbf{z}_t'\mathbf{R}\mathbf{z}_t + 2\mathbf{u}_t'\mathbf{S}\mathbf{z}_t$, and the criterion respects the terminal condition, $\boldsymbol{\mathcal{V}}_T = 0$. As in the Markovian LEQG problem $\mathbf{u}_t$ and $\mathbf{z}_t$ respect a linear Markovian plant equation with a normally distributed error vector, $\boldsymbol{\epsilon}_t$. Under perfect state observation, $\mathbf{z}_t$ is observable at time t and hence c_t is deterministic. This implies that the recursive minimization criterion is well defined.

With respect to the formulation of the criterion proposed by Hansen and Sargent we move the discount factor inside the exponential function (so that rather than using $\delta \ln(E[\exp(\boldsymbol{\mathcal{X}})])$ we employ $\ln(E[\exp(\delta\boldsymbol{\mathcal{X}})])$). By doing this we can easily transform the criterion $\boldsymbol{\mathcal{V}}_t$ into a formulation which resembles that presented in Section 1. In fact, with some manipulations we can write that

$$\boldsymbol{\mathcal{V}}_t \quad = \quad \ln\left(\min_{\mathbf{u}_t}\left\{E_t\left[\exp\left(\frac{\rho}{2}(c_t + \delta\,\boldsymbol{\mathcal{V}}_{t+1})\right)\right]\right\}\right). \tag{3.2}$$

Since the criterion is a quadratic function of the state vector there exists a positive definite quadratic form $Q_t(\mathbf{z}_t, \mathbf{u}_t, \boldsymbol{\epsilon}_{t+1})$ such that $Q_t(\mathbf{z}_t, \mathbf{u}_t, \boldsymbol{\epsilon}_{t+1}) = c_t + \delta\,\boldsymbol{\mathcal{V}}_{t+1}$.[13] Then, we can introduce a modified (or *discounted*) stress function which is equal to $\boldsymbol{\mathcal{S}}_t \equiv c_t - \frac{1}{\rho}\,d_{t+1} + \delta\boldsymbol{\mathcal{V}}_{t+1}$. This differs from stress function proposed by Whittle in two respects: firstly, it covers only periods t and $t+1$; secondly, period $t+1$ criterion is pre-multiplied by the discount factor δ. This formulation for the discounted stress function is appropriate because, under perfect state observation, we can write

$$\begin{aligned}\min_{\mathbf{u}_t} E_t\left[\exp\left(\frac{\rho}{2}(c_t + \delta\,\boldsymbol{\mathcal{V}}_{t+1})\right)\right] \quad &\propto \quad \min_{\mathbf{u}_t}\int \exp\left(\frac{\rho}{2}(c_t + \delta\,\boldsymbol{\mathcal{V}}_{t+1}) - \frac{1}{2}\,\boldsymbol{\epsilon}_{t+1}'\mathbf{N}^{-1}\boldsymbol{\epsilon}_{t+1}\right) d\boldsymbol{\epsilon}_{t+1} \\ &= \quad \min_{\mathbf{u}_t}\int \exp\left(\rho\,\frac{\boldsymbol{\mathcal{S}}_t}{2}\right) d\boldsymbol{\epsilon}_{t+1}\,.\end{aligned}$$

Because the discounted stress function is positive definite in $\mathbf{u}_t$ and negative definite in $\boldsymbol{\epsilon}_t$, restating the properties of the exponential function we find that

$$\min_{\mathbf{u}_t} E_t\left[\exp\left(\frac{\rho}{2}(c_t + \delta\,\boldsymbol{\mathcal{V}}_{t+1})\right)\right] \quad \propto \quad \exp\left(\frac{\rho}{2}\min_{\mathbf{u}_t}\max_{\boldsymbol{\epsilon}_{t+1}}\boldsymbol{\mathcal{S}}_t\right).$$

This indicates that a simplified version of the Risk-sensitive CEP formulated in Theorem 1 applies: to determine the optimal control rule the discounted stress function is now minimized

[13] Under these assumptions the criterion $\boldsymbol{\mathcal{V}}_t$ is a function of the state vector $\mathbf{z}_t$. Following Hansen and Sargent (1994), it can be shown that this function is monotonically increasing and convex in $\mathbf{z}_t$.

with respect to $\mathbf{u}_t$ and maximized with respect to $\boldsymbol{\epsilon}_{t+1}$. In other words, in the perfect state observation scenario the stress function is extremized only with respect to the current control vector, $\mathbf{u}_t$, and next period unobservable, $\boldsymbol{\epsilon}_{t+1}$. Indeed, we can simply solve a modified (discounted) future stress recursion

$$F_t(\mathbf{z}_t) \quad = \quad \min_{\mathbf{u}_t} \left\{ \max_{\boldsymbol{\epsilon}_{t+1}} \left[c_t \, - \, \frac{1}{\rho} \, d_{t+1} \, + \, \delta F_{t+1}(\mathbf{z}_{t+1}) \right] \right\} , \tag{3.3}$$

where $F_{t+1}(\mathbf{z}_{t+1}) = \mathbf{z}'_{t+1} \boldsymbol{\Pi}_{t+1} \mathbf{z}_{t+1}$, which differs from the standard recursion only for the presence of the discount factor, δ, in front of the period $t+1$ extremized future stress function, $F_{t+1}(\mathbf{z}_{t+1})$.[14] Straightforward calculations show that, given the Markovian linear plant equation and the quadratic cost function in the control and state vectors, Theorem 3 must be revised, in that: i) the second order condition for the extremization of the discounted future stress function requires that $\delta \boldsymbol{\Pi}_{t+1} - (1/\rho)\mathbf{N}^{-1}$ is negative definite; and ii) in the modified Riccati equation $\widetilde{\boldsymbol{\Pi}}_{t+1} = ((\delta \boldsymbol{\Pi}_{t+1})^{-1} - \rho \mathbf{N})^{-1}$.

Because of time-discounting it is possible to consider the limit case for $T \uparrow \infty$, that is a LEQG problem with time-discounting and infinite horizon. As indicated by Hansen and Sargent (1994) there is no certainty that for $T \uparrow \infty$ the criterion $\boldsymbol{\mathcal{V}}_t$ is finite and hence the LEQG may be not well defined. However, when a minimum is reached we can identify a stationary solution, in that in the limit $\boldsymbol{\Pi}_t \to \boldsymbol{\Pi}$ and $\mathbf{K}_t \to \mathbf{K}$, where the limit matrices are determined by the fixed point in the risk-sensitive Riccati equation,

$$\begin{aligned} \boldsymbol{\Pi} \quad &= \quad \mathbf{R} \, + \, \mathbf{A}' \widetilde{\boldsymbol{\Pi}} \mathbf{A} \, - \, (\mathbf{S}' \, + \, \mathbf{A}' \widetilde{\boldsymbol{\Pi}} \mathbf{B})(\mathbf{Q} \, + \, \mathbf{B}' \widetilde{\boldsymbol{\Pi}} \mathbf{B})^{-1} (\mathbf{S} \, + \, \mathbf{B}' \widetilde{\boldsymbol{\Pi}} \mathbf{A}) \, , \\ &\text{with} \quad \widetilde{\boldsymbol{\Pi}} \, \equiv \, ((\delta \boldsymbol{\Pi})^{-1} - \rho \mathbf{N})^{-1} \, . \end{aligned}$$

3.2 The Recursive Criterion Under Imperfect State Observation

Hansen and Sargent do not consider the scenario in which only a noisy signal on the state vector $\mathbf{z}_t$ is observed at time t.[15] In such a scenario the initial recursive criterion proposed in equation (3.1) is not well defined, as the cost function c_t is no longer deterministic. However, we can employ the recursive criterion in equation (3.2). Under imperfect state observation, while no longer equivalent to the former one, this criterion is well defined.

[14] Then, the criterion $\boldsymbol{\mathcal{V}}_t$ is such that $\exp(\boldsymbol{\mathcal{V}}_t) = \exp(\frac{1}{2}\rho[\kappa_t + F_t(\mathbf{z}_t)])$, with κ_t independent of $\mathbf{z}_t$.

[15] Hansen and Sargent, in deriving recursive linear control rules for their minimization criterion, rely on results developed by Jacobson (Jacobson, 1973, 1977) to analyze LEQG problems under perfect state observation.

As now $\mathbf{z}_{t-1}$, $\mathbf{z}_t$ and $\mathbf{z}_{t+1}$ are unobservable vectors, the discounted stress function takes a new formulation. For $\boldsymbol{\psi}'_t$ denoting the vector $(\boldsymbol{\epsilon}_t \ \boldsymbol{\eta}_t)'$, $\boldsymbol{\Omega}_{t-1}$ the conditional covariance matrix for $\mathbf{z}_{t-1}$ given the information contained in observation history $\mathbf{H}_{t-1}$ and

$$\mathbf{P} = \begin{pmatrix} \mathbf{N} & \mathbf{L}' \\ \mathbf{L} & \mathbf{M} \end{pmatrix},$$

the discounted stress function is $\boldsymbol{\mathcal{S}}_t \equiv c_t - \frac{1}{\rho}(\boldsymbol{\mathcal{D}}_{t-1} + d_t + d_{t+1}) + \delta \boldsymbol{\mathcal{V}}_{t+1}$, where d_t and d_{t+1} are quadratic forms in $\boldsymbol{\psi}'_t$ and $\boldsymbol{\psi}'_{t+1}$ ($d_t = \boldsymbol{\psi}'_t \mathbf{P}^{-1} \boldsymbol{\psi}_t$ and $d_{t+1} = \boldsymbol{\psi}'_{t+1} \mathbf{P}^{-1} \boldsymbol{\psi}_{t+1}$), while $\boldsymbol{\mathcal{D}}_{t-1}$ is a quadratic form in the vector $\mathbf{z}_{t-1}$ and its conditional expectation $\hat{\mathbf{z}}_{t-1}$ ($\boldsymbol{\mathcal{D}}_{t-1} = (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})' \boldsymbol{\Omega}_{t-1}^{-1} (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})$). In fact, let $\boldsymbol{\Upsilon}_{t-1}$ denote the covariance matrix, conditional on observation history $\mathbf{H}_{t-1}$, for $\boldsymbol{\xi}_t$, where $\boldsymbol{\xi}'_t \equiv (\mathbf{z}'_{t-1} - \hat{\mathbf{z}}'_{t-1} \ \boldsymbol{\psi}'_t \ \boldsymbol{\psi}'_{t+1})$. We immediately notice that

$$\min_{\mathbf{u}_t} E_t \left[\exp\left(\frac{\rho}{2} (c_t + \delta \boldsymbol{\mathcal{V}}_{t+1}) \right) \right] \quad \propto \quad \min_{\mathbf{u}_t} \int \exp\left(\frac{\rho}{2} (c_t + \delta \boldsymbol{\mathcal{V}}_{t+1}) - \frac{1}{2} \boldsymbol{\xi}'_t \boldsymbol{\Upsilon}_{t-1}^{-1} \boldsymbol{\xi}_t \right) d\boldsymbol{\xi}_t .$$

Considering that $(\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})' \perp \boldsymbol{\psi}'_t \perp \boldsymbol{\psi}'_{t+1}$, we can write

$$\begin{aligned} \min_{\mathbf{u}_t} E_t \left[\exp\left(\frac{\rho}{2} (c_t + \delta \boldsymbol{\mathcal{V}}_{t+1}) \right) \right] \quad &\propto \quad \min_{\mathbf{u}_t} \int \exp\left(\frac{\rho}{2} (c_t + \delta \boldsymbol{\mathcal{V}}_{t+1}) - \frac{1}{2} \left(\boldsymbol{\psi}'_{t+1} \mathbf{P}^{-1} \boldsymbol{\psi}_{t+1} \ + \right. \right. \\ &\qquad \left. \left. \boldsymbol{\psi}'_t \mathbf{P}^{-1} \boldsymbol{\psi}_t \ + \ (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})' \boldsymbol{\Omega}_{t-1}^{-1} (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1}) \right) \right) d\boldsymbol{\xi}_t \\ &= \quad \min_{\mathbf{u}_t} \int \exp\left(\rho \frac{\boldsymbol{\mathcal{S}}_t}{2} \right) d\boldsymbol{\xi}_t . \end{aligned}$$

Because the discounted stress function is positive definite in $\mathbf{u}_t$ and negative definite in $\boldsymbol{\xi}_t$, we find that exploiting the properties of the exponential function

$$\min_{\mathbf{u}_t} E_t \left[\exp\left(\frac{\rho}{2} (c_t + \delta \boldsymbol{\mathcal{V}}_{t+1}) \right) \right] \quad \propto \quad \exp\left(\frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\boldsymbol{\xi}_t} \boldsymbol{\mathcal{S}}_t \right) .$$

This implies that the simplified version of the Risk-sensitive CEP which holds with time-discounting must be reformulated under imperfect state observation: $\boldsymbol{\mathcal{S}}_t$ is now minimized with respect to $\mathbf{u}_t$ and maximized with respect to the noise vector $\boldsymbol{\xi}_t$.

Once again, the extremization of the discounted stress can be achieved by splitting $\boldsymbol{\mathcal{S}}_t$ into two components related to past and future. In particular, re-formulating the risk-sensitive SP stated in Theorem 2, the extremization of the discounted stress function can be achieved by:

i) solving the discounted future stress recursion and the past stress recursion, now given by

$$F_t(\mathbf{z}_t) = \min_{\mathbf{u}_t} \left\{ \max_{\mathbf{z}_{t+1}, \mathbf{w}_{t+1}} \left[c_t - \frac{1}{\rho} d_{t+1} + \delta F_{t+1}(\mathbf{z}_{t+1}) \right] \right\}, \tag{3.4}$$

$$P_t(\mathbf{z}_t, \mathbf{H}_t) = \max_{\mathbf{z}_{t-1}} \left[-\frac{1}{\rho} d_t + \boldsymbol{\mathcal{D}}_{t-1} \right], \tag{3.5}$$

and; ii) maximizing $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t)$ with respect to $\mathbf{z}_t$.

4 Monetary Policy for a Risk-averse Central Bank

We now apply this new formulation of the LEQG problem with time-discounting to the issue of output and inflation stabilization for monetary policy. In particular, we refer to a simple analytical framework developed by Svensson (Svennson, 1997) which describes the optimal monetary policy of a central bank with infinite horizon, time-separable quadratic cost function of inflation and output gap. We investigate an extension of Svensson's analysis to the scenario where monetary authorities: i) face a risk-sensitive minimization criterion; and ii) observe imperfectly inflation and output.[16] This allows to see what happens when the CEP cannot be applied as in the LQG problem investigated by Svensson and the actual values of the state variables cannot be replaced by their expectations when they are imperfectly observed.

Svensson studies the optimal policy of a central bank which controls the short-term real interest rate to minimize the expected value of the risk-neutral criterion $\boldsymbol{\mathcal{L}}_t \equiv \sum_{i=0}^{\infty} \delta^i c_{t+i}$, where the cost c_t is quadratic in the inflation rate, π_t, and the output gap y_t, $c_t \equiv \pi_t^2 + \lambda y_t^2$.[17,18] The dynamics of the state variables, π_t and y_t, is given by the following system of linear equations

$$\begin{aligned} \pi_t &= \pi_{t-1} + \alpha y_{t-1} + \epsilon_t^{\pi}, \\ y_t &= \beta y_{t-1} - \gamma r_{t-1} + \epsilon_t^{y}, \end{aligned}$$

where r_t is a short-term interest rate and the coefficients α, β, γ and λ are non-negative

[16] van der Ploeg (2004) also analyzes such an extension. However, he does not allow for time-discounting and hence does not rely on a recursive criterion as the one presented in equation (3.2).

[17] The long-run natural output level is normalized to zero so that y_t corresponds to output gap.

[18] The cost function should depend on the deviation of the inflation rate from a target level π^*. To accommodate this constant term into the cost function see the adjustment to the recursive solution derived by Whittle (See Section 7.5, Whittle, 1990). We abstract from such complication.

constants. The variation in the inflation rate is increasing in lagged output, while the latter is serially correlated and decreasing in the lagged real interest rate. As noted by Svensson the short-term interest rate affects output with one lag and the inflation rate with two lags, this discrepancy being an important feature of this model which is however consistent with ample empirical evidence.

As in the plant equation the error terms ϵ_t^{π} and ϵ_t^{y} follow independent white noise processes, Svensson investigates a standard LQG optimization problem, which we can recast into the LEQG framework with time-discounting by introducing the criterion $\boldsymbol{\mathcal{V}}_t$ and by defining the vector of state variables $\mathbf{z}_t \equiv (\pi_t \ y_t)'$, the vector of error terms $\boldsymbol{\epsilon}_t \equiv (\epsilon_t^{\pi} \ \epsilon_t^{y})'$ and the scalar control variable $u_t \equiv r_t$. We then have that

$$\mathbf{A} \equiv \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}, \ \mathbf{B} \equiv \begin{pmatrix} 0 \\ -\gamma \end{pmatrix}, \ \mathbf{R} \equiv \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \ \mathbf{Q} \equiv \mathbf{S} \equiv \mathbf{0}, \ \mathbf{N} \equiv \begin{pmatrix} \sigma_{\pi}^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}.$$

As the optimization horizon is infinite we concentrate on the steady-state solution: by solving the fixed point for the modified Riccati equation we find that $\exp(\boldsymbol{\mathcal{V}}_t) = \exp(\frac{1}{2}\rho[\kappa + \mathbf{z}_t'\boldsymbol{\Pi}\mathbf{z}_t])$ and $\hat{u}_t = \mathbf{K}'\mathbf{z}_t$, where κ is a constant independent of $\mathbf{z}_t$,

$$\boldsymbol{\Pi} \ = \ \begin{pmatrix} 1 + \delta\, W & \alpha\delta W \\ \alpha\delta W & \lambda + \alpha^2\delta W \end{pmatrix},$$

W is a positive root of the following quadratic equation

$$\delta\left(\alpha^2 \ - \ \delta(\alpha^2+\lambda)\rho\sigma_{\pi}^2\right)W^2 \ - \ \left(\delta(\alpha^2+\lambda) \ - \ \lambda(1 \ - \ \delta\rho\sigma_{\pi}^2)\right)W - \lambda \ = \ 0$$

and

$$\mathbf{K} \ = \ \frac{1}{\gamma}\left(\frac{\alpha\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_{\pi}^2} \quad \beta \ + \ \frac{\alpha^2\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_{\pi}^2}\right),$$

where $\theta = \delta(\lambda + \delta(\alpha^2+\lambda)W)$.This implies that the optimal monetary policy is reached by setting the short-term interest rate equal to

$$r_t \ = \ \frac{1}{\gamma}\,\frac{\alpha\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_{\pi}^2}\,\pi_t \ + \ \frac{1}{\gamma}\left(\beta \ + \ \frac{\alpha^2\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_{\pi}^2}\right) y_t\,, \tag{4.1}$$

a Taylor's rule which clearly subsumes that derived by Svensson for $\rho = 0$, in that under

risk-neutrality,

$$W \;=\; \frac{1}{2}\left(1 \;-\; \frac{(1-\delta)\lambda}{\alpha^2\delta} \;+\; \sqrt{\left(1 \;+\; \frac{(1-\delta)}{\alpha^2\delta}\right)^2 \;+\; \frac{4\,\lambda}{\alpha^2}}\right)$$

and

$$r_t \quad = \quad \frac{1}{\gamma}\,\frac{\alpha\delta W}{\alpha^2\delta W+\lambda}\,\pi_t \;+\; \frac{1}{\gamma}\left(\beta \;+\; \frac{\alpha^2\delta W}{\alpha^2\delta W+\lambda}\right) y_t\,.$$

It is interesting to see that the similarities with Svensson's solution do not exhaust here. In particular, denoting with $\pi_{t+1|t}$ the expectation at time t of the inflation rate in period $t+1$, we have that $\pi_{t+1|t} = \pi_t + \alpha y_t$. It is immediate to verify that

$$r_t \quad = \quad \frac{1}{\gamma}\left(\beta\, y_t \;+\; \frac{\alpha\delta W}{\alpha^2\delta W+\lambda-\theta\rho\sigma_\pi^2}\;\pi_{t+1|t}\right)$$

and that

$$\exp(\boldsymbol{\mathcal{V}}_t) \;=\; \exp\!\left(\frac{1}{2}\rho[\kappa \;+\; \pi_t^2 \;+\; \lambda\, y_t^2 \;+\; \delta\, W\pi_{t+1|t}^2]\right),$$

so that the control path and the minimization criterion can be defined in terms of the inflation forecast. In addition, denoting with $\pi_{t+2|t}$ the expectation at time t of the inflation rate in period $t+2$, we find that at the optimum

$$\pi_{t+2|t} \;=\; -\,\frac{1}{\alpha\delta W}\;\left(\lambda \;-\; \theta\,\rho\,\sigma_\pi^2\right)\; y_{t+1|t}\,,$$

where $y_{t+1|t}$ denotes the expectation at time t of period $t+1$ output gap. This condition implies that the two-period ahead inflation forecast is equal to its target level insofar the one-period ahead expected output gap is null, confirming Svensson's result that with output stabilization the inflation forecast adjusts slowly to the target level.

However, significant differences also emerge between the risk-neutral and risk-averse scenarios. When $\lambda = 0$ and hence only inflation targeting motivates the monetary authorities, the expectation of the inflation rate π_{t+2} in period t is always null for $\rho = 0$. As suggested by Svensson, in the risk-neutral scenario the inflation forecast becomes an explicit intermediate target, in that the monetary policy is optimal insofar $\pi_{t+2|t} = 0$. On the other hand, for $\rho > 0$ at the optimum $\pi_{t+2|t} = \alpha\delta\rho\sigma_\pi^2\, y_{t+1|t}$. Because of risk-aversion, the standard CEP cannot be applied as in the LQG problem investigated by Svensson and hence the inflation forecast is

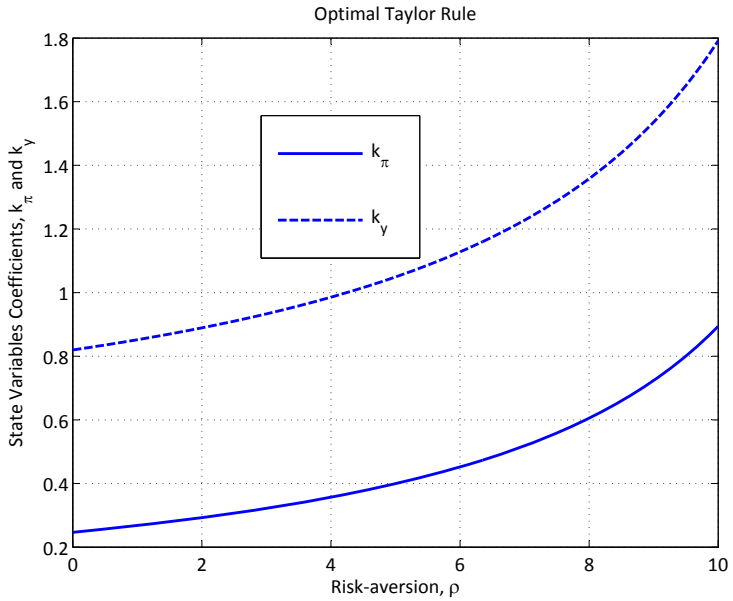

Figure 3: The values of the state variable coefficients k_π and k_y for $\alpha = 1.5$, $\beta = 0.9$, $\delta = 0.95$, $\gamma = 2$, $\lambda = 1$ and $\sigma_\pi^2 = \sigma_\pi^2 = 0.05$ against the risk-aversion coefficient ρ.

not longer an explicit intermediate target when inflation targeting is the mission of the central bank.

In addition, even when the monetary policy is also aimed at output stabilization ($\lambda > 0$), in the risk-neutral scenario we see that inflation forecasts dampen out, in that for $\rho = 0$ $\pi_{t+2|t} = \frac{\lambda}{\alpha^2 \delta W + \lambda} \pi_{t+1|t}$. This indicates that the central bank expects the inflation rate to reach the target level in the long-run. This is not necessarily the case for $\rho > 0$. Strikingly, the central bank may actually expect the inflation rate to wander away from the target level even if λ is small or null, that is even if output stabilization is not the main objective of its monetary policy. In fact, for $\lambda = 0$ we see that $\pi_{t+2|t} = \frac{-\delta\rho\sigma_\pi^2}{1-\delta\rho\sigma_\pi^2} \pi_{t+1|t}$, and hence for $1/2 < \delta\rho\sigma_\pi^2 < 1$ $\text{abs}(\pi_{t+2|t}) > \text{abs}(\pi_{t+1|t})$.

Finally, risk-aversion conditions deeply the Taylor rule selected by the monetary autorithies. Figure 3 plots the inflation, k_π, and output gap, k_y, coefficients in the optimal Taylor rule described in equation (4.1), against the risk-aversion coefficient for values of ρ ranging from 0 to 10.[19] As $\rho = 0$ corresponds to risk-neutrality we see that a risk-averse central bank

[19]These coefficients are determined by: solving for the positive root of the quadratic equation which pins down W; and ii) inserting the resulting value in the vector $\mathbf{K}$.

follows a more aggressive Taylor rule, in that the real short-term interest rate is more sensitive to: i) departures of the inflation rate from its target level; and ii) swings in output from full employment. While Figure 3 is obtained for a specific choice of parameters, numerical analysis shows that such a result holds for all the parametric constellations for which an optimal monetary policy exists.

4.1 Imperfect State Observation

It is interesting to see what happens in the case the central bank observes imperfectly the state variables. In the LQG case we know that thanks to the CEP it is sufficient to replace the state vector with its ML estimate. This is not the case when the central bank is risk-averse as the unobservable variables are replaced by those values which maximize the discounted stress function.

With respect to the monetary policy of the central bank, a realistic scenario is that in which the monetary authorities observe the state variables with one lag. In this scenario the formulation of the discounted stress function is simplified, in that $d_t = \boldsymbol{\epsilon}_t' \mathbf{N}^{-1} \boldsymbol{\epsilon}_t$. Then, the solution of the past stress recursion is straightforward in that a maximum in (3.5) is reached for $\mathbf{z}_{t-1} = \hat{\mathbf{z}}_{t-1}$ and is given by $P_t(\mathbf{z}_t, \mathbf{H}_t) = -\frac{1}{\rho}\, \boldsymbol{\epsilon}_t' \mathbf{N}^{-1} \boldsymbol{\epsilon}_t + \cdots$, where once again $+\cdots$ denotes terms independent of $\mathbf{z}_t$. Since $F_t(\mathbf{z}_t) = \mathbf{z}_t' \boldsymbol{\Pi}_t \mathbf{z}_t$, in re-coupling past and future extemization we solve

$$\max_{\mathbf{z}_t} \left\{ -\frac{1}{\rho}\, \boldsymbol{\epsilon}_t' \mathbf{N}^{-1} \boldsymbol{\epsilon}_t + \mathbf{z}_t' \boldsymbol{\Pi}_t \mathbf{z}_t \right\} .$$

Considering that if $\hat{\mathbf{z}}_t$ is the conditional expectation of the state vector at time t,[20] we can write $\boldsymbol{\epsilon}_t = \mathbf{z}_t - \hat{\mathbf{z}}_t$, so that we need to solve

$$\max_{\mathbf{z}_t} \left\{ -\frac{1}{\rho}\, (\mathbf{z}_t - \hat{\mathbf{z}}_t)' \mathbf{N}^{-1} (\mathbf{z}_t - \hat{\mathbf{z}}_t) + \mathbf{z}_t' \boldsymbol{\Pi}_t \mathbf{z}_t \right\} .$$

We immediately conclude that the maximum total stress estimate (MTSE) $\breve{\mathbf{z}}_t$ is given by

$$\breve{\mathbf{z}}_t \;=\; (\mathbf{I} \,-\, \rho \mathbf{N}\, \boldsymbol{\Pi}_t)^{-1} \hat{\mathbf{z}}_t \,.$$

As indicated in Theorem 2, the optimal control is obtained by inserting the MTSE, $\breve{\mathbf{z}}_t$, into the control rule which would prevail under perfect state observation. Within the monetary policy example we find that in equation (4.1) the actual values of the inflation rate and output

[20] Given that the observable vector is now $\mathbf{w}_t = \mathbf{z}_{t-1}$, $\hat{\mathbf{z}}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}\mathbf{u}_{t-1}$.

gap are not replaced by their ML estimates, but by the following values which correct for the impact of risk-aversion

$$\breve{\pi}_t = \hat{\pi}_t + \left(\frac{\boldsymbol{\pi}_1 - \det(\boldsymbol{\Pi})\rho\sigma_y^2}{\det(\mathbf{I}_2 - \delta\rho\boldsymbol{\Pi})}\,\hat{\pi}_t + \frac{\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2 - \delta\rho\boldsymbol{\Pi})}\hat{y}_t \right) \rho\,\sigma_\pi^2$$

$$\breve{y}_t = \hat{y}_t + \left(\frac{\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2 - \delta\rho\boldsymbol{\Pi})}\,\hat{\pi}_t + \frac{\boldsymbol{\pi}_2 - \det(\boldsymbol{\Pi})\rho\sigma_\pi^2}{\det(\mathbf{I}_2 - \delta\rho\boldsymbol{\Pi})}\hat{y}_t \right) \rho\,\sigma_y^2\,,$$

where $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_{1,2}$ and $\boldsymbol{\pi}_2$ are the elements of matrix $\boldsymbol{\Pi}$.

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Appendix: Detailed Calculations

Future Stress Recursion. Consider the solution of the future stress recursion. First we need to solve

$$\max_{\boldsymbol{\epsilon}_{n+1}} \left[c_n \, - \, \frac{1}{\rho} \, d_{n+1} \, + \, F_{n+1}(\mathbf{z}_{n+1}) \right] ,$$

where $d_{n+1} \equiv \boldsymbol{\epsilon}'_{n+1} \mathbf{N}_{n+1}^{-1} \boldsymbol{\epsilon}_{n+1}$, $c_n \equiv 2(\mathbf{z}_n + \boldsymbol{\Lambda}_n \, \mathbf{u}_n + \boldsymbol{\epsilon}_{n+1})' \mathbf{u}_n$ and $F_{n+1}(\mathbf{z}_{n+1}) \equiv \mathbf{z}'_{n+1} \boldsymbol{\Pi}_{n+1} \mathbf{z}_{n+1}$. The first order condition is

$$2 \, \mathbf{u}_n \, - \, \frac{2}{\rho} \, \mathbf{N}_{n+1}^{-1} \, \boldsymbol{\epsilon}_{n+1} \; + \; 2 \, \boldsymbol{\Pi}_{n+1} \, (\mathbf{z}_n \, + \, \boldsymbol{\Lambda}_n \, \mathbf{u}_n \, + \, \boldsymbol{\epsilon}_{n+1}) \; = \; \mathbf{0} \, ,$$

so that

$$\begin{aligned}
\hat{\boldsymbol{\epsilon}}_{n+1} \quad &= \quad - \left(\boldsymbol{\Pi}_{n+1} - \frac{1}{\rho} \mathbf{N}_{n+1}^{-1} \right)^{-1} \left(\mathbf{u}_n + \boldsymbol{\Pi}_{n+1} (\mathbf{z}_n \, + \, \boldsymbol{\Lambda}_n \, \mathbf{u}_n) \right) \\
&= \quad - \left(\boldsymbol{\Pi}_{n+1} - \frac{1}{\rho} \mathbf{N}_{n+1}^{-1} \right)^{-1} \boldsymbol{\Pi}_{n+1} \left(\mathbf{z}_n \, + \, \widetilde{\boldsymbol{\Lambda}}_n \, \mathbf{u}_n \right) \\
&= \quad \rho \, \mathbf{N}_{n+1} \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1} \left(\mathbf{z}_n \, + \, \widetilde{\boldsymbol{\Lambda}}_n \, \mathbf{u}_n \right) ,
\end{aligned}$$

as $(\boldsymbol{\Pi}_{n+1} - \frac{1}{\rho} \mathbf{N}_{n+1}^{-1})^{-1} = -\rho \, \mathbf{N}_{n+1} (\boldsymbol{\Pi}_{n+1}^{-1} - \rho \mathbf{N}_{n+1})^{-1} \boldsymbol{\Pi}_{n+1}^{-1}$.[21] Inserting this expression into the future stress recursion we minimize with respect to $\mathbf{u}_n$

$$\begin{aligned}
&2 \mathbf{u}'_n \mathbf{z}_n + 2 \mathbf{u}'_n \boldsymbol{\Lambda}_n \mathbf{u}_n + 2 \rho \mathbf{u}'_n \mathbf{N}_{n+1} \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1} \left(\mathbf{z}_n + \widetilde{\boldsymbol{\Lambda}}_n \mathbf{u}_n \right) + \\
&- \rho \left(\mathbf{z}_n + \widetilde{\boldsymbol{\Lambda}}_n \mathbf{u}_n \right)' \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1} \mathbf{N}_{n+1} \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1} \left(\mathbf{z}_n + \widetilde{\boldsymbol{\Lambda}}_n \mathbf{u}_n \right) + \\
&\left((\mathbf{I}_m + \rho \mathbf{N}_{n+1} \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1}) \left(\mathbf{z}_n + \widetilde{\boldsymbol{\Lambda}}_n \mathbf{u}_n \right) - \boldsymbol{\Pi}_{n+1}^{-1} \mathbf{u}_n \right)' \boldsymbol{\Pi}_{n+1} \left((\mathbf{I}_m + \rho \mathbf{N}_{n+1} \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1}) \left(\mathbf{z}_n + \widetilde{\boldsymbol{\Lambda}}_n \mathbf{u}_n \right) - \boldsymbol{\Pi}_{n+1}^{-1} \mathbf{u}_n \right) ,
\end{aligned}$$

where we have used the fact that $\mathbf{z}_n + \boldsymbol{\Lambda}_n \mathbf{u}_n + \hat{\boldsymbol{\epsilon}}_{n+1} = (\mathbf{I}_m + \rho \mathbf{N}_{n+1} \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1})(\mathbf{z}_n + \widetilde{\boldsymbol{\Lambda}}_n \mathbf{u}_n) - \boldsymbol{\Pi}_{n+1}^{-1} \mathbf{u}_n$. Rearranging this is equivalent to minimize with respect to $\mathbf{u}_n$ the following expression

$$-\mathbf{u}'_n \boldsymbol{\Pi}_{n+1}^{-1} \mathbf{u}_n + (\mathbf{z}_n + \widetilde{\boldsymbol{\Lambda}}_n \mathbf{u}_n)' \left((\mathbf{I}_m + \rho \mathbf{N}_{n+1} \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1})' \boldsymbol{\Pi}_{n+1} (\mathbf{I}_m + \rho \mathbf{N}_{n+1} \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1}) - \rho \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1} \mathbf{N}_{n+1} \widetilde{\boldsymbol{\Pi}}_{n+1}^{-1} \right) (\mathbf{z}_n + \widetilde{\boldsymbol{\Lambda}}_n \mathbf{u}_n) .$$

[21] This holds because if $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices, then $(\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1} (\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1} \mathbf{B}^{-1} = \mathbf{B}^{-1} (\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1} \mathbf{A}^{-1}$.

Now it can be shown that $(\mathbf{I}_m + \rho\mathbf{N}_{n+1}\widetilde{\mathbf{\Pi}}_{n+1}^{-1})$ is equal to $\mathbf{\Pi}_{n+1}^{-1}\widetilde{\mathbf{\Pi}}_{n+1}^{-1}$. In fact, multiplying by $\widetilde{\mathbf{\Pi}}_{n+1}$ on both sides, we find that $\widetilde{\mathbf{\Pi}}_{n+1} + \rho\mathbf{N}_{n+1} = \mathbf{\Pi}_{n+1}^{-1}$, which corresponds to the definition of $\widetilde{\mathbf{\Pi}}_{n+1}$. Then,

$$\left((\mathbf{I}_m + \rho\mathbf{N}_{n+1}\widetilde{\mathbf{\Pi}}_{n+1}^{-1})'\mathbf{\Pi}_{n+1}(\mathbf{I}_m + \rho\mathbf{N}_{n+1}\widetilde{\mathbf{\Pi}}_{n+1}^{-1}) - \rho\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\mathbf{N}_{n+1}\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\right) \;=\; \widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,.$$

Hence, we need to minimize with respect to $\mathbf{u}_n$

$$-\mathbf{u}_n'\mathbf{\Pi}_{n+1}^{-1}\mathbf{u}_n + (\mathbf{z}_n + \widetilde{\mathbf{\Lambda}}_n\mathbf{u}_n)'\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,(\mathbf{z}_n + \widetilde{\mathbf{\Lambda}}_n\mathbf{u}_n)\,.$$

The first order condition is then

$$-2\,\mathbf{\Pi}_{n+1}^{-1}\mathbf{u}_n \;+\; 2\,\widetilde{\mathbf{\Lambda}}_n'\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,\widetilde{\mathbf{\Lambda}}_n\,\mathbf{u}_n \;+\; 2\,\widetilde{\mathbf{\Lambda}}_n\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,\mathbf{z}_n \;=\; \mathbf{0}\,,$$

from which equation (2.9) immediately ensues that, under the second order condition $-\mathbf{\Pi}_{n+1}^{-1} + \widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n' > 0$, the optimal control is given by

$$\hat{\mathbf{u}}_n \;=\; -\,\boldsymbol{\mathcal{B}}_n\,\mathbf{z}_n\,, \qquad \text{where} \quad \boldsymbol{\mathcal{B}}_n \;=\; \left(-\,\mathbf{\Pi}_{n+1}^{-1} \;+\; \widetilde{\mathbf{\Lambda}}_n\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,\widetilde{\mathbf{\Lambda}}_n'\right)^{-1}\,\widetilde{\mathbf{\Lambda}}_n\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,.$$

This implies that $\mathbf{z}_n + \widetilde{\mathbf{\Lambda}}_n\hat{\mathbf{u}}_n = (\mathbf{I}_m - \widetilde{\mathbf{\Lambda}}_n\boldsymbol{\mathcal{B}}_n)\,\mathbf{z}_n$. Therefore, the extremized future stress in auction n is equal to

$$F_n(\mathbf{z}_n) \;=\; -\,\mathbf{z}_n'\,\boldsymbol{\mathcal{B}}_n'\mathbf{\Pi}_{n+1}^{-1}\boldsymbol{\mathcal{B}}_n\mathbf{z}_n \;+\; \mathbf{z}_n'(\mathbf{I}_m - \widetilde{\mathbf{\Lambda}}_n\boldsymbol{\mathcal{B}}_n)'\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,(\mathbf{I}_m - \widetilde{\mathbf{\Lambda}}_n\boldsymbol{\mathcal{B}}_n)\,\mathbf{z}_n\,.$$

This means that $F_n(\mathbf{z}_n) = \mathbf{z}_n'\mathbf{\Pi}_n\mathbf{z}_n$, where, defining $\mathbf{\Phi}_n = (-\,\mathbf{\Pi}_{n+1}^{-1} \;+\; \widetilde{\mathbf{\Lambda}}_n\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,\widetilde{\mathbf{\Lambda}}_n')^{-1}$,

$$\begin{aligned}
\mathbf{\Pi}_n \;&=\; -\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,\widetilde{\mathbf{\Lambda}}_n'\,\mathbf{\Phi}_n'\mathbf{\Pi}_{n+1}^{-1}\mathbf{\Phi}_n\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1} \;+\; \left(\mathbf{I}_m \;-\; \widetilde{\mathbf{\Lambda}}_n'\,\mathbf{\Phi}_n\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\right)'\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\left(\mathbf{I}_m \;-\; \widetilde{\mathbf{\Lambda}}_n'\,\mathbf{\Phi}_n\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\right)\\
&=\; -\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\,\widetilde{\mathbf{\Lambda}}_n'\,\mathbf{\Phi}_n'\mathbf{\Pi}_{n+1}^{-1}\mathbf{\Phi}_n\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1} \;+\; \widetilde{\mathbf{\Pi}}_{n+1}^{-1} - \;2\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n'\,\mathbf{\Phi}_n\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\\
&\quad + \widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n'\,\mathbf{\Phi}_n'\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n'\,\mathbf{\Phi}_n\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\\
&=\; \widetilde{\mathbf{\Pi}}_{n+1}^{-1} \;-\; 2\,\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n'\,\mathbf{\Phi}_n\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1} \;+\; \widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n'\,\mathbf{\Phi}_n'\underbrace{\left(\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n' \;-\; \mathbf{\Pi}_{n+1}^{-1}\right)}_{-\mathbf{\Phi}_n'^{-1}}\mathbf{\Phi}_n\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\\
&=\; \widetilde{\mathbf{\Pi}}_{n+1}^{-1} \;-\; \widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n'\,\mathbf{\Phi}_n\,\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}
\end{aligned}$$

as indicated in equation (2.10).

Single Risky Asset Scenario. For $m = 1$ assuming that $\mathbf{\Pi}_{n+1}$ is written as $-2\alpha_n$ we find that

$$\widetilde{\mathbf{\Pi}}_{n+1}^{-1} \;=\; -\frac{2\alpha_n}{1 + 2\alpha_n\lambda_n^2\rho\sigma_l^2}$$

while

$$\widetilde{\mathbf{\Lambda}}_n \;=\; -\,\frac{1-2\alpha_n\lambda_n}{2\alpha_n}$$

so that

$$\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1} \;=\; \frac{1-2\alpha_n\lambda_n}{1+2\alpha_n\lambda_n^2\rho\sigma_l^2} \quad \text{and} \quad \widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n' \;=\; -\frac{1}{2\alpha_n}\,\frac{(1-2\alpha_n\lambda_n)^2}{1+2\alpha_n\lambda_n^2\rho\sigma_l^2}\,.$$

This means that

$$-\,\mathbf{\Pi}_{n+1}^{-1} \;+\; \widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n' \;=\; \frac{2\lambda_n(1-\alpha_n\lambda_n) \,+\, \lambda_n^2\rho\sigma_n^2}{1+2\alpha_n\lambda_n^2\rho\sigma_n^2}$$

so that: i) the condition $-\,\mathbf{\Pi}_{n+1}^{-1}+\widetilde{\mathbf{\Lambda}}_n\widetilde{\mathbf{\Pi}}_{n+1}^{-1}\widetilde{\mathbf{\Lambda}}_n' > 0$ corresponds to $2\lambda_n(1-\alpha_n\lambda_n) + \lambda_n^2\rho\sigma_n^2 > 0$; and ii)

$$\beta_n \;=\; \frac{1-2\alpha_n\lambda_n}{2\lambda_n(1-\alpha_n\lambda_n) \,+\, \lambda_n^2\rho\sigma_n^2}\,.$$

Finally,

$$\begin{aligned}
-2\alpha_{n-1} \;&=\; -\,\frac{2\alpha_n}{1+2\alpha_n\lambda_n^2\rho\sigma_l^2} \;-\; \frac{(1-2\alpha_n\lambda_n)^2}{(1+2\alpha_n\lambda_n^2\rho\sigma_l^2)\,(2\lambda_n(1-\alpha_n\lambda_n) \,+\, \lambda_n^2\rho\sigma_n^2)} \\
&=\; -\,\frac{1}{(1+2\alpha_n\lambda_n^2\rho\sigma_l^2)\,(2\lambda_n(1-\alpha_n\lambda_n) \,+\, \lambda_n^2\rho\sigma_n^2)}\left(1-4\alpha_n\lambda_n+4\alpha_n^2\lambda_n^2+2\alpha_n\lambda_n^2\rho\sigma_l^2+4\alpha_n\lambda_n-4\alpha_n^2\lambda_n^2\right) \\
&=\; -\,\frac{1}{2\lambda_n(1-\alpha_n\lambda_n) \,+\, \lambda_n^2\rho\sigma_n^2}\,.
\end{aligned}$$

Recursive Solution of the System of Difference Equations. Consider that

$$\lambda_n \;=\; \frac{\beta_n\Sigma_{n-1}}{\beta_n^2\Sigma_{n-1}+\sigma_l^2\Delta n}\,,$$

while $\Sigma_n = (1-\lambda_n\beta_n)\Sigma_{n-1}$. Substituting the former in the latter we see that

$$\Sigma_n \;=\; \frac{\sigma_l^2\Delta n}{\beta_n^2\Sigma_{n-1}+\sigma_l^2\Delta n}\,\Sigma_{n-1}\,,$$

so that

$$\lambda_n \;=\; \frac{\beta_n}{\sigma_l^2\Delta_n}\,\frac{\sigma_l^2\Delta_n}{\beta_n^2\Sigma_{n-1}+\sigma_l^2\Delta n}\Sigma_{n-1} \;=\; \frac{\beta_n\Sigma_n}{\sigma_l^2\Delta_n}\,.$$

Replacing the expression for β_n into λ_n we find that

$$\sigma_l^2\Delta_n\lambda_n \;=\; \frac{(1-2\alpha\lambda_n)\Sigma_n}{2\lambda_n(1-\alpha\lambda_n)+\rho\sigma_l^2\Delta_n\lambda_n^2}\,,$$

which is equivalent to

$$\Sigma_n \;+\rho\sigma_l^2\Delta_n^2\lambda_n^2 \;=\; 2(1-\alpha_n\lambda_n)(\Sigma_n-\sigma_l^2\Delta_n\lambda_n^2)\,.$$

This equation possesses three roots, one negative and two positive. Both the negative root and the

larger of the two positive roots do not respect the condition $2\lambda_n(1-\alpha_n\lambda_n) + \lambda_n^2\rho\sigma_n^2 > 0$. Thus, given Σ_n and α_n, λ_n is uniquely determined. From this β_n, Σ_{n-1} and α_{n-1} are then derived. Hence, given a starting value for Σ_N and the terminal condition $\alpha_N = 0$, the numerical function $\Sigma' = G(\Sigma_N)$ is established.

Symmetry of Liquidity Matrix, $\mathbf{\Lambda}_N$. Consider that for $n = N$ $\boldsymbol{\mathcal{B}}_N = (2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1})^{-1}$. Inserting this expression into (2.13) we find that

$$\mathbf{\Lambda}_N \quad = \quad \mathbf{\Sigma}_{N-1}\left(2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1}\right)^{'-1}\left((2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1})^{-1}\mathbf{\Sigma}_{N-1}(2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1})^{'-1} + \mathbf{\Sigma}_l\Delta_N\right)^{-1}.$$

This can be written as

$$\begin{aligned}\mathbf{\Lambda}_N \quad = \quad & \mathbf{\Sigma}_{N-1}\left(2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1}\right)^{'-1}(2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1})'\mathbf{\Sigma}_{N-1}^{-1}(2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1}) \times \\ & \left((2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1})'\mathbf{\Sigma}_{N-1}^{-1}(2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1}) + \mathbf{\Sigma}_l^{-1}\Delta_N^{-1}\right)^{-1}\mathbf{\Sigma}_l^{-1}\Delta_N^{-1},\end{aligned}$$

so that

$$\mathbf{\Lambda}_N\mathbf{\Sigma}_l\Delta_N\ \left((2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1})'\mathbf{\Sigma}_{N-1}^{-1}(2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1}) + \mathbf{\Sigma}_l^{-1}\Delta_N^{-1}\right) \quad = \quad 2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1}.$$

This is equivalent to

$$\mathbf{\Lambda}_N\mathbf{\Sigma}_l\Delta_N\ (2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1})'\mathbf{\Sigma}_{N-1}^{-1}(2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1}) \quad = \quad \mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1}, \quad \text{i.e.}$$

$$(2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1})'\mathbf{\Sigma}_{N-1}^{-1}(2\mathbf{\Lambda}_N + \rho\mathbf{N}_{N+1}) \quad = \quad \mathbf{\Sigma}_l^{-1}\Delta_N^{-1}\ (\mathbf{I}_m + \rho\mathbf{\Lambda}_N^{-1}\mathbf{N}_{N+1}).$$

Since $\mathbf{N}_{N+1} = \mathbf{\Lambda}_N\mathbf{\Sigma}_l\Delta_N\mathbf{\Lambda}_N'$ we conclude that

$$(2\mathbf{I}_m + \rho\mathbf{\Lambda}_N\mathbf{\Sigma}_l\Delta_N)\mathbf{\Lambda}_N'\mathbf{\Sigma}_{N-1}^{-1}\mathbf{\Lambda}_N(2\mathbf{I}_m + \rho\mathbf{\Sigma}_l\Delta_N\mathbf{\Lambda}_N') \quad = \quad \mathbf{\Sigma}_l^{-1}\Delta_N^{-1}\ (\mathbf{I}_m + \rho\mathbf{\Sigma}_l\Delta_N\mathbf{\Lambda}_N'), \quad \text{that is}$$

$$\boldsymbol{\mathcal{M}}_N'\mathbf{\Lambda}_N'\mathbf{\Sigma}_{N-1}^{-1}\,\mathbf{\Lambda}_N\,\boldsymbol{\mathcal{M}}_N \quad = \quad \mathbf{\Sigma}_l^{-1}\Delta_N^{-1}\,(\boldsymbol{\mathcal{M}}_N - \mathbf{I}_m),$$

where $\boldsymbol{\mathcal{M}}_N \equiv 2\mathbf{I}_m + \rho\mathbf{\Sigma}_l\Delta_N\mathbf{\Lambda}_N'$.

Minimization Criterion with Time-Discounting. Under perfect state observation we can write that

$$\begin{aligned}
\exp\left(\boldsymbol{\mathcal{V}}_t\right) &= \exp\left(\min_{\mathbf{u}_t}\left\{\frac{\rho}{2}c_t \;+\; \ln\left(E_t\left[\exp\left(\delta\,\frac{\rho}{2}\,\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right\}\right) \\
&= \min_{\mathbf{u}_t}\left\{\exp\left(\frac{\rho}{2}c_t \;+\; \ln\left(E_t\left[\exp\left(\delta\,\frac{\rho}{2}\,\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right)\right\} \\
&= \min_{\mathbf{u}_t}\left\{\exp\left(\frac{\rho}{2}c_t\right)\cdot\exp\left(\ln\left(E_t\left[\exp\left(\delta\,\frac{\rho}{2}\,\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right)\right\} \\
&= \min_{\mathbf{u}_t}\left\{\exp\left(\frac{\rho}{2}c_t\right)\cdot\left(E_t\left[\exp\left(\delta\,\frac{\rho}{2}\,\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right\} \\
&= \min_{\mathbf{u}_t}\left\{E_t\left[\exp\left(\frac{\rho}{2}(c_t \;+\; \delta\,\boldsymbol{\mathcal{V}}_{t+1})\right)\right]\right\},
\end{aligned}$$

so that

$$\boldsymbol{\mathcal{V}}_t \;=\; \ln\left(\min_{\mathbf{u}_t}\left\{E_t\left[\exp\left(\frac{\rho}{2}(c_t \;+\; \delta\,\boldsymbol{\mathcal{V}}_{t+1})\right)\right]\right\}\right).$$

Modified Stress Function Recursion. Recall that if $Q(\mathbf{u},\boldsymbol{\epsilon})$ is a quadratic form which is positive definite in $\boldsymbol{\epsilon}$ and negative definite in $\mathbf{u}$ then the following holds

$$\min_{\mathbf{u}}\int\exp\left[-\frac{1}{2}\,Q(\mathbf{u},\boldsymbol{\epsilon})\,d\boldsymbol{\epsilon}\right] \;\propto\; \exp\left[-\frac{1}{2}\,\max_{\mathbf{u}}\min_{\boldsymbol{\epsilon}}\,Q(\mathbf{u},\boldsymbol{\epsilon})\right].$$

Hence, consider that under perfect state observation

$$\begin{aligned}
\min_{\mathbf{u}_t}\int\exp\left(\rho\frac{\boldsymbol{\mathcal{S}}_t}{2}\right)d\boldsymbol{\epsilon}_{t+1} &= \min_{\mathbf{u}_t}\int\exp\left(-\frac{1}{2}\,\underbrace{(-\rho\boldsymbol{\mathcal{S}}_t)}_{Q(\mathbf{u}_t,\boldsymbol{\epsilon}_{t+1})}\right)d\boldsymbol{\epsilon}_{t+1} \\
&\propto \exp\left(-\frac{1}{2}\,\max_{\mathbf{u}_t}\min_{\boldsymbol{\epsilon}_{t+1}}(-\rho\boldsymbol{\mathcal{S}}_t)\right) \;=\; \exp\left(\frac{\rho}{2}\,\min_{\mathbf{u}_t}\max_{\boldsymbol{\epsilon}_{t+1}}\boldsymbol{\mathcal{S}}_t\right),
\end{aligned}$$

where we have made use of the fact that $\boldsymbol{\mathcal{S}}_t$ is positive definite in $\mathbf{u}_t$ and negative definite in $\boldsymbol{\epsilon}_{t+1}$ so that the opposite holds for $-\boldsymbol{\mathcal{S}}_t$. With a similar argument, replacing $\boldsymbol{\epsilon}_{t+1}$ with $\boldsymbol{\xi}_t$, we show that under imperfect state observation

$$\min_{\mathbf{u}_t}\int\exp\left(\rho\frac{\boldsymbol{\mathcal{S}}_t}{2}\right)d\boldsymbol{\xi}_t \;\propto\; \exp\left(\frac{\rho}{2}\,\min_{\mathbf{u}_t}\max_{\boldsymbol{\xi}_t}\boldsymbol{\mathcal{S}}_t\right).$$

Future Stress Recursion with Time-Discounting. Both in the standard LEQG problem and in the LEQG problem with time-discounting the future stress function $F_t(\mathbf{z}_t)$ respects the recursion

$$F_t \;=\; \boldsymbol{\mathcal{L}}\,\tilde{\boldsymbol{\mathcal{L}}}\,F_{t+1},$$

based on the following two operators

$$\boldsymbol{\mathcal{L}}\,\phi(\mathbf{z}) \;=\; \min_{\mathbf{u}}\,[c(\mathbf{z},\mathbf{u}) \;+\; \phi(A\mathbf{z}+B\mathbf{u})] \quad \text{and} \quad \tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) \;=\; \max_{\boldsymbol{\epsilon}}\,[\phi(\mathbf{z}+\boldsymbol{\epsilon}) \;-\; \frac{1}{\rho}\boldsymbol{\epsilon}'\mathbf{N}^{-1}\boldsymbol{\epsilon}]\,.$$

In the future stress recursion for the standard LEQG formulation $\phi(\mathbf{z}) = \mathbf{z}'\boldsymbol{\Pi}\mathbf{z}$, so that

$$\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) \;=\; \max_{\boldsymbol{\epsilon}}\,[(\mathbf{z}+\boldsymbol{\epsilon})'\boldsymbol{\Pi}(\mathbf{z}+\boldsymbol{\epsilon}) \;-\; \frac{1}{\rho}\boldsymbol{\epsilon}'\mathbf{N}^{-1}\boldsymbol{\epsilon}]\,.$$

Taking first derivatives, we find that

$$\hat{\boldsymbol{\epsilon}} \;=\; -(\boldsymbol{\Pi} \;-\; \frac{1}{\rho}\mathbf{N}^{-1})^{-1}\boldsymbol{\Pi}\,\mathbf{z} \;=\; -\breve{\boldsymbol{\Pi}}^{-1}\boldsymbol{\Pi}\,\boldsymbol{z}\,.$$

Replacing this expression we conclude that $\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) \;=\; \mathbf{z}'(\boldsymbol{\Pi}^{-1} \;-\; \rho\mathbf{N})^{-1}\mathbf{z} \;=\; \mathbf{z}'\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{z}$. The second order condition entais that to have a maximum matrix $\breve{\boldsymbol{\Pi}}$ must be negative definite. For $\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) = \mathbf{z}'\widetilde{\boldsymbol{\Pi}}\,\mathbf{z}$, solution of the operator $\boldsymbol{\mathcal{L}}$ yields the standard recursive formulae from the LQG problem where $\widetilde{\boldsymbol{\Pi}} = (\boldsymbol{\Pi}^{-1} \;-\; \rho\mathbf{N})^{-1}$ replaces $\boldsymbol{\Pi}$.

In the future stress recursion for the LEQG problem with time-discounting $\phi(\mathbf{z})$ is equal to $\delta\mathbf{z}'\boldsymbol{\Pi}\mathbf{z} = \mathbf{z}'\delta\boldsymbol{\Pi}\mathbf{z}$, so that $\delta\boldsymbol{\Pi}$ simply replaces $\boldsymbol{\Pi}$. This means that $\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) \;=\; \mathbf{z}'\widetilde{\boldsymbol{\Pi}}\mathbf{z}$, where now $\widetilde{\boldsymbol{\Pi}} = (\delta\boldsymbol{\Pi}^{-1} - \rho\mathbf{N})^{-1}$ and $\breve{\boldsymbol{\Pi}} \;=\; \delta\boldsymbol{\Pi} - \frac{1}{\rho}\mathbf{N}^{-1}$. Solution of the operator $\boldsymbol{\mathcal{L}}$ is unaffected by time-discounting, in that the standard recursive formulae from the LQG problem now apply with $\widetilde{\boldsymbol{\Pi}} = (\delta\boldsymbol{\Pi}^{-1} \;-\; \rho\mathbf{N})^{-1}$ replacing $\boldsymbol{\Pi}$.

Optimal Monetary Policy. In the stationary solution,

$$\begin{aligned}
\widetilde{\boldsymbol{\Pi}} \;&=\; (\delta\,\boldsymbol{\Pi}^{-1} \;-\; \rho\mathbf{N})^{-1} \;=\; \delta\boldsymbol{\Pi}\,(\mathbf{I}_2 \;-\; \delta\,\rho\,\mathbf{N}\,\boldsymbol{\Pi})^{-1} \\
&=\; \delta\boldsymbol{\Pi}\begin{pmatrix} 1-\delta\rho\sigma_\pi^2\boldsymbol{\pi}_1 & -\delta\rho\sigma_\pi^2\boldsymbol{\pi}_{1,2} \\ -\delta\rho\sigma_y^2\boldsymbol{\pi}_{1,2} & 1-\delta\rho\sigma_y^2\boldsymbol{\pi}_2 \end{pmatrix}^{-1} \\
&=\; \frac{\delta}{\det(\mathbf{I}_2-\delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})}\begin{pmatrix} \boldsymbol{\pi}_1 & \boldsymbol{\pi}_{1,2} \\ \boldsymbol{\pi}_{1,2} & \boldsymbol{\pi}_2 \end{pmatrix}\begin{pmatrix} 1-\delta\rho\sigma_y^2\boldsymbol{\pi}_2 & \delta\rho\sigma_\pi^2\boldsymbol{\pi}_{1,2} \\ \delta\rho\sigma_y^2\boldsymbol{\pi}_{1,2} & 1-\delta\rho\sigma_\pi^2\boldsymbol{\pi}_1 \end{pmatrix} \\
&=\; \frac{\delta}{\det(\mathbf{I}_2-\delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})}\begin{pmatrix} (1-\delta\rho\sigma_y^2\boldsymbol{\pi}_2)\boldsymbol{\pi}_1 \;+\; \delta\rho\sigma_y^2\boldsymbol{\pi}_{1,2}^2 & \boldsymbol{\pi}_{1,2} \\ \boldsymbol{\pi}_{1,2} & (1-\delta\rho\sigma_\pi^2\boldsymbol{\pi}_1)\boldsymbol{\pi}_2 \;+\; \delta\rho\sigma_\pi^2\boldsymbol{\pi}_{1,2}^2 \end{pmatrix} \\
&=\; \frac{\delta}{\det(\mathbf{I}_2-\delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})}\,\widehat{\boldsymbol{\Pi}}\,.
\end{aligned}$$

where

$$\det(\mathbf{I}_2-\delta\,\rho\mathbf{N}\,\boldsymbol{\Pi}) \;\;=\;\; 1-\delta\rho(\sigma_\pi^2\boldsymbol{\pi}_1 \;+\; \sigma_y^2\boldsymbol{\pi}_2) \;+\; \delta^2\,\rho^2\det(\boldsymbol{\Pi})\,\sigma_\pi^2\sigma_y^2\,.$$

It is immediate to check that $\mathbf{B}'\widehat{\mathbf{\Pi}}\mathbf{B} = \gamma^2\,\hat{\boldsymbol{\pi}}_2$, so that

$$(\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1} \;=\; \frac{1}{\delta}\,\frac{1}{\gamma^2}\,\frac{1}{\hat{\boldsymbol{\pi}}_2}\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})\,, \qquad \mathbf{B}(\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}' \;=\; \det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi}) \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\delta}\,\frac{1}{\hat{\boldsymbol{\pi}}_2} \end{pmatrix}.$$

Hence,

$$\mathbf{B}(\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\,\widetilde{\mathbf{\Pi}} \;=\; \begin{pmatrix} 0 & 0 \\ \frac{1}{\delta}\,\frac{\hat{\boldsymbol{\pi}}_{1,2}}{\hat{\boldsymbol{\pi}}_2} & 1 \end{pmatrix}, \qquad \mathbf{I}_2 \;-\; \mathbf{B}(\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\,\widetilde{\mathbf{\Pi}} \;=\; \begin{pmatrix} 1 & 0 \\ -\frac{1}{\delta}\,\frac{\hat{\boldsymbol{\pi}}_{1,2}}{\hat{\boldsymbol{\pi}}_2} & 0 \end{pmatrix}.$$

In the modified Riccati equation we have

$$\begin{aligned}
\mathbf{\Pi} \;&=\; \mathbf{R} \;+\mathbf{A}'\widetilde{\mathbf{\Pi}}\left(\mathbf{I}_2 \;-\; \mathbf{B}(\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\,\widetilde{\mathbf{\Pi}}\right)\mathbf{A} \\
&=\; \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \;+\; \frac{\delta}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\pi}}_1 & \hat{\boldsymbol{\pi}}_{1,2} \\ \hat{\boldsymbol{\pi}}_{1,2} & \hat{\boldsymbol{\pi}}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\hat{\boldsymbol{\pi}}_{1,2}}{\hat{\boldsymbol{\pi}}_2} & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \\
&=\; \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \;+\; \frac{\delta}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \frac{\det(\hat{\mathbf{\Pi}})}{\hat{\boldsymbol{\pi}}_2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \\
&=\; \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \;+\; \delta\,\frac{\det(\hat{\mathbf{\Pi}})}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})}\,\frac{1}{\hat{\boldsymbol{\pi}}_2} \begin{pmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{pmatrix}.
\end{aligned}$$

Then we can define $W = \frac{1}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})}\left(\hat{\boldsymbol{\pi}}_1 - \frac{\hat{\boldsymbol{\pi}}_{1,2}^2}{\hat{\boldsymbol{\pi}}_2}\right)$ and conclude that

$$\boldsymbol{\pi}_1 \;=\; 1 \;+\; \delta\,W\,, \quad \boldsymbol{\pi}_{1,2} \;=\; \alpha\,\delta\,W\,, \quad \boldsymbol{\pi}_2 \;=\; \lambda \;+\; \alpha^2\,\delta\,W\,.$$

Now,

$$\begin{aligned}
\hat{\boldsymbol{\pi}}_1 - \frac{\hat{\boldsymbol{\pi}}_{1,2}^2}{\hat{\boldsymbol{\pi}}_2} \;&=\; \boldsymbol{\pi}_1 \;-\; \delta\,\rho\det(\mathbf{\Pi})\,\sigma_y^2 \;-\frac{\boldsymbol{\pi}_{1,2}^2}{\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2} \\
&=\; \frac{(\boldsymbol{\pi}_1 \;-\; \delta\,\rho\det(\mathbf{\Pi})\,\sigma_y^2)\,(\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2) - \boldsymbol{\pi}_{1,2}^2}{\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2} \\
&=\; \frac{\det(\mathbf{\Pi})\left[1 - \delta\rho(\sigma_\pi^2\boldsymbol{\pi}_1 \;+\; \sigma_y^2\boldsymbol{\pi}_2) \;+\; \delta^2\,\rho^2\det(\mathbf{\Pi})\,\sigma_\pi^2\sigma_y^2\right]}{\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2} \\
&=\; \frac{\det(\mathbf{\Pi})\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})}{\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2}\,,
\end{aligned}$$

so that $W = \frac{\det(\mathbf{\Pi})}{\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2}$. Given the expressions for $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_{1,2}$ and $\boldsymbol{\pi}_2$, we have that $\det(\mathbf{\Pi}) =$

$\lambda + \delta(\alpha^2 + \lambda)W$, so that

$$W \;=\; \frac{\lambda \;+\; \delta\,(\alpha^2+\lambda)\,W}{\lambda\,(1-\delta\rho\sigma_\pi^2) \;+\; \delta\Big(\alpha^2 \;-\; \delta\,(\alpha^2+\lambda)\,\rho\,\sigma_\pi^2\Big)\,W}\,.$$

Rearranging we find that

$$\delta\left(\alpha^2 \;-\; \delta(\alpha^2+\lambda)\rho\sigma_\pi^2\right)W^2 \;-\; \left(\delta(\alpha^2+\lambda) \;-\; \lambda \;+\; \delta\lambda\rho\sigma_\pi^2)\right)W - \lambda \;=\; 0$$

whose roots are

$$W^{\pm} \quad = \quad \frac{\delta(\alpha^2+\lambda) \;-\; \lambda\,(1-\delta\rho\sigma_\pi^2) \;\pm\; \Delta^{1/2}}{2\,\delta\Big(\alpha^2 \;-\; \delta(\alpha^2+\lambda)\rho\sigma_\pi^2\Big)} \quad \text{where}$$

$\Delta = \Big(\delta(\alpha^2+\lambda) - \lambda(1-\delta\rho\sigma_\pi^2)\Big)^2 + 4\delta\lambda(\alpha^2 - \delta(\alpha^2+\lambda)\rho\sigma_\pi^2\Big)$. For $\rho = 0$, $\Delta = \Big(\delta\alpha^2 - (1-\delta)\lambda)\Big)^2 + 4\alpha^2\delta\lambda$, while

$$W^{\pm} \quad = \quad \frac{1}{2}\left(1 \;-\; \frac{(1-\delta)\lambda \pm \Delta^{1/2}}{\alpha^2\delta}\right) \;=\; = \;\frac{1}{2}\left(1 \;-\; \frac{(1-\delta)\lambda}{\alpha^2\delta} \;\pm\; \sqrt{\left(1 \;+\; \frac{(1-\delta)}{\alpha^2\delta}\right)^2 \;+\; \frac{4\,\lambda}{\alpha^2}}\right).$$

To determine $\mathbf{K}$ consider that

$$\begin{aligned}
\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{A} \quad &= \quad \frac{\delta}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})}(0 \quad -\gamma)\begin{pmatrix} \hat{\boldsymbol{\pi}}_1 & \hat{\boldsymbol{\pi}}_{1,2} \\ \hat{\boldsymbol{\pi}}_{1,2} & \hat{\boldsymbol{\pi}}_2 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \\
&= \quad -\frac{\delta\,\gamma}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})}\;(\hat{\boldsymbol{\pi}}_{1,2} \quad \alpha\hat{\boldsymbol{\pi}}_{1,2} \;+\; \beta\hat{\boldsymbol{\pi}}_2)\;.
\end{aligned}$$

Given that $\mathbf{K} = -(\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{A}$, we find that

$$\mathbf{K} \quad = \quad \frac{1}{\gamma}\left(\frac{\hat{\boldsymbol{\pi}}_{1,2}}{\hat{\boldsymbol{\pi}}_2} \quad \alpha\frac{\hat{\boldsymbol{\pi}}_{1,2}}{\hat{\boldsymbol{\pi}}_2} \;+\; \beta\right).$$

Finally, since $\hat{\boldsymbol{\pi}}_{1,2} = \boldsymbol{\pi}_{1,2} = \alpha\delta W$ and $\hat{\boldsymbol{\pi}}_2 = \boldsymbol{\pi}_2 - \delta\det(\mathbf{\Pi})\rho\sigma_\pi^2 = \lambda + \alpha^2\delta W - \delta(\lambda + \delta(\alpha^2+\lambda)W)\rho\sigma_\pi^2$, we find that

$$\mathbf{K} \quad = \quad \frac{1}{\gamma}\left(\frac{\alpha\delta W}{\alpha^2\delta W + \lambda - \delta(\lambda + \delta(\alpha^2+\lambda)W)\rho\sigma_\pi^2} \qquad \beta \;+\; \frac{\alpha^2\delta W}{\alpha^2\delta W + \lambda - \delta(\lambda + \delta(\alpha^2+\lambda)W)\rho\sigma_\pi^2}\right).$$

To reach a minimum $\delta\mathbf{\Pi}_{t+1} - (1/\rho)\mathbf{N}^{-1}$ must be negative definite. This corresponds to the double

condition that

$$\delta\boldsymbol{\pi}_1 - \frac{1}{\rho}\frac{1}{\sigma_\pi^2} \;<\; 0\,, \quad (\delta\boldsymbol{\pi}_1 - \frac{1}{\rho}\frac{1}{\sigma_\pi^2})\,(\delta\boldsymbol{\pi}_2 - \frac{1}{\rho}\frac{1}{\sigma_y^2}) \;-\; \delta^2\,\boldsymbol{\pi}_{1,2} \;>\; 0\,.$$

The Value Function and the Inflation Forecast. Given the plant equation for π_t we immediately see that $\pi_{t+1|t} = \pi_t + \alpha y_t$. Then, consider that

$$\begin{aligned}
\mathbf{z}_t'\boldsymbol{\Pi}\mathbf{z}_t &= (\pi_t\ y_t)\begin{pmatrix} 1+\delta W & \alpha\delta W \\ \alpha+\delta W & \lambda+\alpha^2\delta W \end{pmatrix}\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} \\
&= (\pi_t\ y_t)\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} + (\pi_t\ y_t)\begin{pmatrix} \delta W & \alpha\delta W \\ \alpha\delta W & \alpha^2\delta W \end{pmatrix}\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} \\
&= \pi_t^2 + \lambda y_t^2 + \delta\, W\,(\pi_t\ y_t)\begin{pmatrix} 1 \\ \alpha \end{pmatrix}\begin{pmatrix} 1 & \alpha \end{pmatrix}\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} \\
&= \pi_t^2 + \lambda y_t^2 + \delta\, W\,(\pi_t + \alpha y_t)^2\,.
\end{aligned}$$

Immediately it follows that

$$\exp(\boldsymbol{\mathcal{V}}_t) \;=\; \exp\!\left(\frac{1}{2}\rho[\kappa + \pi_t^2 + \lambda\, y_t^2 + \delta\, W\pi_{t+1|t}^2]\right).$$

Inflation and Output Gap Forecast. Since $\pi_{t+1|t} = \pi_t + \alpha y_t$ we find that

$$r_t \;=\; \frac{1}{\gamma}\left(\beta\, y_t + \frac{\alpha\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2}\ \pi_{t+1|t}\right)$$

Inserting this into the plant equation for output gap, we find that

$$y_{t+1|t} \;=\; -\frac{\alpha\,\delta\ W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2}\pi_{t+1|t}\,.$$

Since $\pi_{t+2|t} = \pi_{t+1|t} + \alpha y_{t+1|t}$ and $\pi_{t+1|t} = -\frac{\alpha^2\delta W+\lambda-\theta\rho\sigma_\pi^2}{\alpha\delta W}\ y_{t+1|t}$, we conclude that

$$\pi_{t+2|t} \;=\; -\frac{1}{\alpha\delta W}\ \left(\lambda - \theta\,\rho\,\sigma_\pi^2\right)\ y_{t+1|t}\,.$$

Optimal Monetary Policy with Imperfect State Observation. In the stationary solution, we find that

$$\breve{\mathbf{z}}_t \;=\; (\mathbf{I}_2 - \rho\mathbf{N}\boldsymbol{\Pi})^{-1}\hat{\mathbf{z}}_t, \quad \text{where}$$

$$(\mathbf{I}_2 \,-\, \rho\mathbf{N}\mathbf{\Pi})^{-1} \;=\; \frac{1}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})} \begin{pmatrix} 1-\rho\sigma_y^2\boldsymbol{\pi}_2 & \rho\sigma_\pi^2\boldsymbol{\pi}_{1,2} \\ \rho\sigma_y^2\boldsymbol{\pi}_{1,2} & 1-\rho\sigma_\pi^2\boldsymbol{\pi}_1 \end{pmatrix},$$

so that

$$\begin{aligned} \breve{\pi}_t &= \left(\frac{1-\rho\sigma_y^2\boldsymbol{\pi}_2}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})}\,\hat{\pi}_t \,+\, \frac{\rho\,\sigma_\pi^2\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})}\hat{y}_t\right), \\ \breve{y}_t &= \left(\frac{\rho\,\sigma_y^2\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})}\,\hat{\pi}_t \,+\, \frac{1-\rho\sigma_\pi^2\boldsymbol{\pi}_1}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})}\hat{y}_t\right). \end{aligned}$$

Given that

$$\begin{aligned} \frac{1-\rho\sigma_y^2\boldsymbol{\pi}_2}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})} &= 1 \,+\, \frac{\boldsymbol{\pi}_1-\det(\mathbf{\Pi})\rho\sigma_y^2}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})}\,\rho\,\sigma_\pi^2\,, \\ \frac{1-\rho\sigma_\pi^2\boldsymbol{\pi}_1}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})} &= 1 \,+\, \frac{\boldsymbol{\pi}_2-\det(\mathbf{\Pi})\rho\sigma_\pi^2}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})}\,\rho\,\sigma_y^2\,, \end{aligned}$$

we conclude that the MTSE is

$$\begin{aligned} \breve{\pi}_t &= \hat{\pi}_t \,+\, \left(\frac{\boldsymbol{\pi}_1-\det(\mathbf{\Pi})\rho\sigma_y^2}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})}\,\hat{\pi}_t \,+\, \frac{\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})}\hat{y}_t\right)\rho\,\sigma_\pi^2\,, \\ \breve{y}_t &= \hat{y}_t \,+\, \left(\frac{\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})}\,\hat{\pi}_t \,+\, \frac{\boldsymbol{\pi}_2-\det(\mathbf{\Pi})\rho\sigma_\pi^2}{\det(\mathbf{I}_2-\delta\rho\mathbf{\Pi})}\hat{y}_t\right)\rho\,\sigma_y^2\,. \end{aligned}$$