

Growth and Crisis, Unavoidable Connection?

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Abstract

Emerging economies can experience periods of rapid growth and large capital inflows, followed by sudden stops in their borrowing ability and consequent default. Recoveries are characterized by lower growth rates and often reversed balance of payments. This pattern is difficult to explain in standard models of debt sustainability with temporary shocks to the productivity. I construct a growth model of small open economy where the simple presence of decreasing marginal returns to capital can generate a default event along the path of economic development. Default entails output costs and temporary autarky. The economy features two stages of growth. The first is characterized by constant marginal returns to capital, while in the second stage marginal returns are decreasing and the economy eventually converges to a steady state. The transition between the two stages is stochastic. In both stages I derive endogenous borrowing limits. A sharp reduction in the level of sustainable debt can occur at the turning point between the two stages. An economy with initially high productivity of capital grows very fast, accumulates a large stock of debt, and defaults at the turning point. An economy with initially lower productivity grows at a slower rate and moves to the second stage without default. High growth, high risk and a volatile path for output can then be linked together. I show that debt can be sustained by reputation for repayment in equilibria where the evolution of the debt has bubble-like characteristics. Permanent reversals in the balance of payment can take place after a default event.

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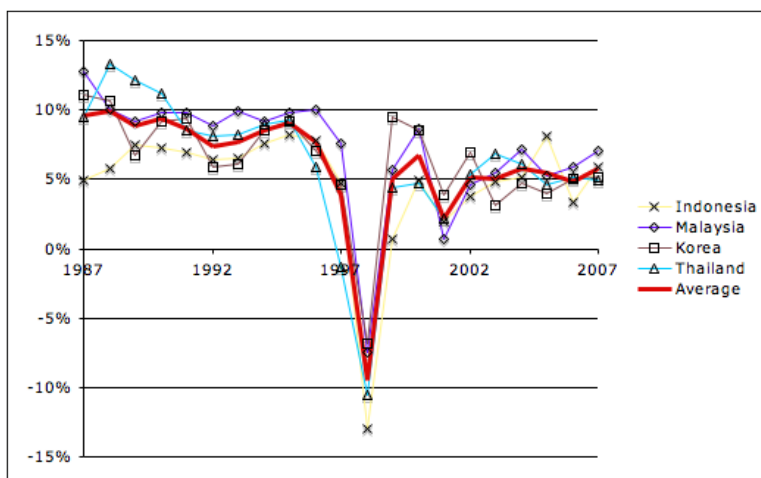


Figure 1: Growth Rates Selected Asian Countries

1 Introduction

This paper analyses the relation between growth and financial crises, here defined as situations where a country defaults on its international debt and, as a consequence, experiences a period of economic distress. As pointed out by Lucas (1990) in his seminal work, theory suggests that, in a world without barriers to international capital movements, capital should flow from developed to developing countries, in order to take advantage of the higher returns that there can be gained. This line of reasoning is based on the assumption that a country is poor because its endowment of capital, relatively to that of other fixed factors, is small. As the economy develops, the marginal return of capital decreases, the capital flow reverses sign since the country starts to pay back its debt with the rest of the world (Barro et al. [1995]). However, both the empirical observation (Lucas [1990]) and the theoretical analysis (Bulow and Rogoff [1989]), emphasize that this type of path for the capital flow is all but granted. Bulow and Rogoff (1989), in particular, show that limited enforceability on debt contracts can play a significant role in the ability of a country to borrow from the rest of the world.

On the empirical side, Ranciere, Tornell and Westermann (2005) point out that we do observe large capital flows from developed to developing countries, but we also observe financial crises and sudden reversals in the balance of payments. Developing economies that have very large growth rates seem

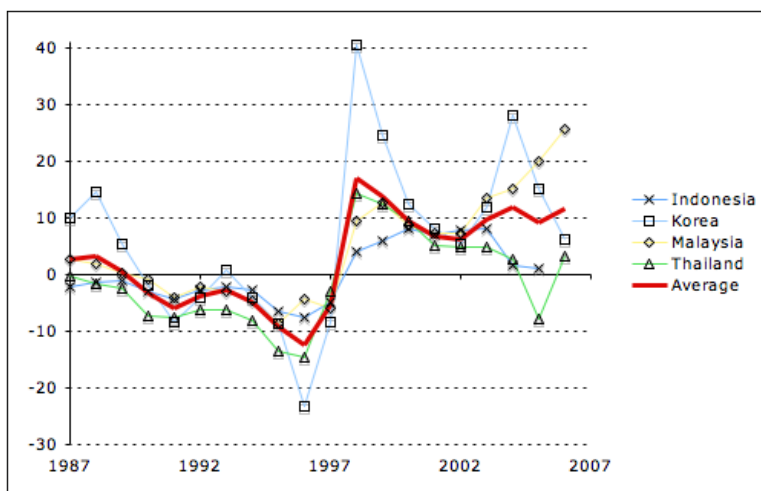


Figure 2: Current Account Selected Asian Countries

also to be more exposed to financial crises.

A striking example of this situation is the case of the Asian crisis of the late '90s, which is considered one of the deepest financial crises of the past 20 years. Figure 1 shows the GDP growth rate for four Asian countries in the years 1987-2007. The year of the crises is clearly marked by a fall of the GDP of about 10% in most of the countries. Foreign capital inflows, shown in Figure 2, that had helped fueling the growth in the first half of the decade, stopped. The year 1998 marks also a permanent shift in the pattern of growth, which was on average 8.1% in the pre-crisis years, and 5% thereafter. A similar pattern is followed by the investment rate presented in Figure 3. Such pre- and post- crisis paths are not specific to the experience of the emerging Asian economies. As Lee and Rhee (2000) and Ranciere, Tornell and Westermann (2005) show, the same regularities are confirmed in a wider sample of countries that have experience financial crises.

The goal of this paper is to explore the relation between growth, debt sustainability and default events. Much of the exiting literature (see for instance Arellano [2008]) have studied country defaults in stationary economies, open to international capital flows, in a context of limited contract enforceability and stochastic shocks affecting output.

The presence of a stochastic environment is a necessary requirement whenever we want to study economies that, in equilibrium, are subject to default. I therefore retain the assumption of a stochastic environment. Moreover, in the spirit of Solow (1954), I assume that the existence of decreasing marginal

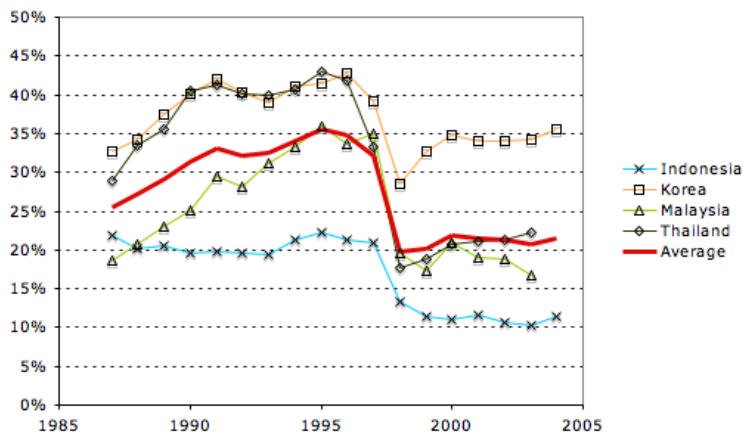


Figure 3: Investment-GDP Ratios Selected Asian Countries

returns to capital eventually lead the growth process of a country to a steady state. However, as a novel element, I place the uncertainty on the time that it takes for the country to converge to such a steady state. In particular, the growth path of the country will feature two stages: in the first stage, marginal returns to capital are constant, while in the second stage they are strictly decreasing over time. The “turning point”, that is the moment when the economy transitions from stage one to stage two, is uncertain. The uncertainty is determined both by some, exogenously given, fundamental and by the speed at which the economy grows during stage one: the faster the growth rate during this stage, the higher the probability that, at each point in time, decreasing marginal returns appear and the economy moves to the second stage of growth. Uncertainty on the fundamental makes also uncertain the level of the steady state GDP, which can be seen as a form of conditional convergence of the growth process.

The country has access to an international financial market. Agents in the economy can choose to default on their debt obligations, and the punishment for default is stochastic autarky during which there is a reduction in productivity. The possibility of default imposes an endogenous constraint on the borrowing ability of the agents. I will show that two paths of development are possible for the country. If the endogenous borrowing limit is sufficiently loose during the first stage of growth the country experiences a large capital inflow (credit boom) which fuels high growth and investment rates. However, when decreasing marginal returns start to appear, the en-

ogenous borrowing limit is sharply reduced and the country experiences a sudden stop in the capital inflow. The consequent credit crunch permanently reduces the growth rate and forces the economy to the default. If the credit crunch is sufficiently strong, the balanced of payment is reversed.

The second possibility is that the the endogenous borrowing limit is very tight during the first stage of growth. The lower availability of credit reduces the growth rate of the economy. However, at the turning point, there is no credit crunch, the economy does not default and moves smoothly to its second stage of development.

The first type of growth path is followed by economies with initially very high marginal product of capital or that face strong punishments for default. Economies that develop with high growth rate are also more unstable. Finally I show that high growth rates can be associated with equilibria where the evolution of international debt resembles a bubble. For these economies I show that, in the context of Bulow and Rogoff (1989), the existence of bubble is strictly linked to debt sustainability.

The remaining of the paper is organized as follows. Section 2.1 presents the problem of the household. Section 2.2 introduces the aggregate technology and defines a general equilibrium for the model. Section 2.3 derives the endogenous borrowing limits and characterizes the equilibrium. Section 2.4 shows the connection between bubbles, debt sustainability and dynamics of the balanced of payments. Section 3 concludes.

2 The Model

2.1 Problem of the Household

A small country is inhabited by a unit measure of infinitely lived identical households. Each household derives utility from a flow of consumption \tilde{c}_t , whose value is

$$E \left[\int_0^{\infty} e^{-\rho t} u(\tilde{c}_t) dt \right]$$

For analytical convenience I take $u(\cdot) = \log \tilde{c}$.

At any time $t \geq 0$ households can issue an amount of debt \tilde{b}_t , bought by competitive international investors, who are risk neutral and discount the future at the constant rate $\rho > 0$. Households can operate a constant return to scale technology which employs two inputs: capital k_t and a composite input h_t . To use an amount h_t of the composite input, the household has to pay a total cost $\tilde{w}_t h_t$ and the price \tilde{w}_t is taken as given. The technology produces at time t a single good y_t that can be used for consumption \tilde{c}_t , for

investment \tilde{x}_t which increases the stock of physical capital, to pay for the cost $\tilde{w}_t h_t$, or for repaying the debt to international investors. The production function for y_t is

$$y_t = \tilde{A} k_t^\alpha h_t^{1-\alpha}$$

Capital is accumulated at the individual level according to the linear technology

$$\dot{k}_t = -\delta k_t + \tilde{x}_t \quad (1)$$

At the end of the instantaneous production process, the net output to the household is then

$$\tilde{A} k_t^\alpha h_t^{1-\alpha} - \tilde{w}_t h_t \quad (2)$$

The stock of debt evolves according to

$$\dot{\tilde{b}}_t = \tilde{r}_t(\tilde{b}_t, k_t) \tilde{b}_t + \tilde{d}_t \quad (3)$$

where \tilde{d}_t is the net new debt issued and $\tilde{r}_t(\tilde{b}_t, k_t)$ is the instantaneous interest rate, which depends on both \tilde{b}_t and k_t . The function $\tilde{r}(\cdot)$ is taken as given by the household.

The household can choose to default on her debt obligations. Default entails exclusion (autarky) from the financial market for a random length of time τ , which has an exponential distribution with mean $\frac{1}{\theta}$, $\theta > 0$. Default is complete, so the household is readmitted to the financial market with a level of outstanding debt equal to zero ¹. During the periods of exclusion, the household suffers a cost which reduces the net output to

$$\xi[\tilde{A} k_t^\alpha h_t^{1-\alpha} - w_t h_t] \quad \xi \in (0, 1) \quad (4)$$

The household faces at each time t a borrowing constraint

$$\tilde{b}_t \leq \tilde{m}_t k_t \quad (5)$$

I call \tilde{m}_t the bound to the maximum *leverage ratio* that the household can attain. The evolution of the leverage bound \tilde{m}_t is exogenous to the household's choice.

There is one element of aggregate uncertainty in the economy. In fact, at a stochastic moment \tilde{t} , the economy is hit by a shock that I call "turning point". The shock, with arrival rate π_0 , has a permanent effect on the price \tilde{w}_t , on the leverage bound m_t , and \tilde{r}

$$\begin{aligned} \tilde{w}(t, \tilde{t}) &= w_0 & \tilde{m}(t, \tilde{t}) &= m_0 & \tilde{r}(\tilde{b}, k, t, \tilde{t}) &= r_0(\tilde{b}, k) & t < \tilde{t} \\ \tilde{w}(t, \tilde{t}) &= w(t - \tilde{t}) & \tilde{m}(t, \tilde{t}) &= m(t - \tilde{t}) & \tilde{r}(\tilde{b}, k, t, \tilde{t}) &= r(\tilde{b}, k, t - \tilde{t}) & t \geq \tilde{t} \end{aligned}$$

¹The model where I allow for the possibility of partial default is presented at the end of Section 2.4.

For some functions $r_0(\cdot)$, $w(\cdot)$, $m(\cdot)$, $r(\cdot)$ continuous in $t - \tilde{t}$ and constants w_0 and m_0 . I assume that, for $\tau \geq 0$,

$$\begin{aligned} m_0 &< M \\ \bar{m}(\tau) &< M \end{aligned}$$

Notice that the path of leverage bound \tilde{m}_t could be discontinuous and have a downwards jump at the turning point. The leverage bounds are smaller than a constant $0 < M < 1$ arbitrarily close to one, and $w_0 > 0$. To gain some intuition for the model, we can think of $w(\tau)$ as an increasing function, so that, after the turning point, the cost \tilde{w}_t of the input h_t keeps rising. As we will see, this creates a situation where the marginal product of capital falls over time. On the contrary, prior to the turning point, the cost \tilde{w}_t , the leverage bound and the interest rate function are constant. In particular, since $\tilde{w}_t = w_0$, the marginal product of capital is also constant. The variable π_0 and the path of w_t and m_t are endogenous to the model. However, they are taken as given by the household. Hence, for the definition of the household's problem, we don't need to explain now what are the general equilibrium effects, presented in Section 2.2 through which they're determined.

The household can choose, at each point in time, whether to default or to repay her debt. We study this choice by considering two strategies: the *repayment* and the *default* strategy. The analysis of these two strategies allows us to derive endogenous leverage bounds m_0 and $m(\tau)$, which represent the maximum level of debt, relative to the capital stock, that the household can credibly commit to repay. The repayment strategy is constructed by having the household repay the debt at any moment, with the exception that default takes place at the arrival of the turning point if and only if $\tilde{b}_{\tilde{t}} > m(0)k_{\tilde{t}}$. In this case, in fact, the leverage ratio of the household is bigger than the maximum supportable leverage bound, and event which is clearly possible only if $m_0 > m(0)$. The default strategy at any time t is constructed by having the household default at t and then follow the repayment strategy after the household is readmitted to the financial market. The default strategy is then a "one stage deviation" from the repayment strategy. Given capital and debt stock, we obtain the values \bar{V} and \bar{V}^d for, respectively, the repayment and default strategy at any time $\tilde{t} + \tau$ after the turning point,²

² \bar{V}'_{τ} is the partial derivative with respect to time that accounts for the change in $\bar{w}(\tau)$ and $r(\tau)$. The total differential with respect to time is then $\frac{d\bar{V}}{dt} = \bar{V}'_k \dot{k} + \bar{V}'_b \dot{b} + \bar{V}'_{\tau} - \rho \bar{V}$. Similar comments hold for the other value functions.

$$\rho \bar{V}(k, \tilde{b}, \tau) = \max_{\tilde{x}, \tilde{d}, h} u(\tilde{c}) + \bar{V}'_k \dot{k} + \bar{V}'_b \dot{\tilde{b}} + \bar{V}'_\tau \quad (6)$$

$$\begin{aligned} \text{s.t. } \quad & \tilde{c} = \tilde{A}k^\alpha h^{1-\alpha} - w(\tau)h_t + \tilde{d} - \tilde{x} \\ & \dot{k} = -\delta k + \tilde{x} \\ & \dot{\tilde{b}} = r(\tau)\tilde{b} + \tilde{d} \\ & \tilde{b} \leq m(\tau)k \\ & \tilde{d} \leq \bar{D}(\tau)k \end{aligned}$$

$$(\rho + \theta) \bar{V}^d(k, \tau) = \max_{\tilde{x}^d, h} u(\tilde{c}^d) + \theta \bar{V}(k, 0, \tau) + \bar{V}'_k{}^d \dot{k} + \bar{V}'_\tau{}^d \quad (7)$$

$$\begin{aligned} \text{s.t. } \quad & \tilde{c}^d = \xi[\tilde{A}k^\alpha h^{1-\alpha} - \bar{w}(\tau)h] - \tilde{x} \\ & \dot{k} = -\delta k + \tilde{x}^d \end{aligned}$$

The values V and V^d from, respectively, the repayment and default strategies before the turning point are

$$(\rho + \pi_0)V(k, \tilde{b}) = \max_{\tilde{x}, \tilde{d}, h} u(\tilde{c}) + \pi_0 \bar{V}^0(k, \tilde{b}) + V'_k \dot{k} + V'_b \dot{\tilde{b}} \quad (8)$$

$$\begin{aligned} \text{s.t. } \quad & \tilde{c} = \tilde{A}k^\alpha h^{1-\alpha} - w_0 h + \tilde{d} - \tilde{x} \\ & \dot{k} = -\delta k + \tilde{x} \\ & \dot{\tilde{b}} = r_0 \tilde{b} + \tilde{d} \\ & \tilde{b} \leq m_0 k \\ & \tilde{d} \leq D_0 k \end{aligned}$$

$$(\rho + \theta + \pi_0)V^d(k) = \max_{\tilde{x}^d, h} u(\tilde{c}^d) + \theta V(k, 0) + \pi_0 \bar{V}^d(k, 0) + V'_k{}^d \dot{k} \quad (9)$$

$$\begin{aligned} \text{s.t. } \quad & \tilde{c}^d = \xi(\tilde{A}k^\alpha h^{1-\alpha} - w_0 h) - \tilde{x}^d \\ & \dot{k} = -\delta k + \tilde{x}^d \end{aligned}$$

The constraints,

$$\tilde{d} \leq \bar{D}(\tau)k, \quad \tilde{d} \leq D_0k, \quad (10)$$

are needed for well defined problems by binding above the control variable \tilde{d} . The value $\bar{V}^0(k, \tilde{b})$ at the turning point $\tilde{t} = t$ depends on whether there is default,

$$\bar{V}^0(k, \tilde{b}) = \begin{cases} \bar{V}(k, \tilde{b}, 0) & \text{if } \tilde{b} \leq m(0)k_t \\ \bar{V}^d(k, 0) & \text{if } \tilde{b} > m(0)k \end{cases} \quad (11)$$

We let the household choose, at any moment, between the repayment and the default strategy, and we define the optimal values V^* and \bar{V}^* as

$$\begin{aligned} \bar{V}^*(k, \tilde{b}, \tau) &= \max\{V(k, \tilde{b}, \tau), V^d(k, \tau)\} \quad t \geq 0 \\ V^*(k, \tilde{b}) &= \max\{V(k, \tilde{b}), V^d(k)\} \end{aligned}$$

We consider only leverage bounds m_0 and $m(\tau)$ that are *sustainable*, i.e. leverage bounds with the property that the repayment strategy is always more valuable than the default strategy, i.e.

$$\tilde{b} \leq m(\tau)k \quad \Rightarrow \quad \bar{V}^*(\tilde{b}, k, \tau) = \bar{V}(k, \tilde{b}, \tau) \quad (12)$$

$$\tilde{b} \leq m_0k \quad \Rightarrow \quad V^*(\tilde{b}, k) = V(\tilde{b}, k) \quad (13)$$

The international lenders are competitive, risk-neutral and discount time at rate ρ . When the leverage bounds are sustainable, the lenders charge the following interest rate. According to the repayment strategy, there is never default after the turning point \tilde{t} , then at any time $t = \tilde{t} + \tau$

$$r(\tau, \tilde{b}, k) = \rho \quad (14)$$

The situation prior to the turning point is somewhat different. The repayment strategy requires default at the turning point \tilde{t} if and only if the leverage ratio of the household at \tilde{t} is strictly greater than $m(0)$. The interest rate is then

$$r_0(\tilde{b}, k) = \begin{cases} \rho & \text{if } \tilde{b} \leq m(0)k \\ \rho + \pi_0 & \text{if } \tilde{b} > m(0)k \end{cases} \quad (15)$$

Households have the same initial conditions at time zero and the leverage bound is initially binding. If we label households by $i \in [0, 1]$ then

$$k_0^i = k_0 \quad \tilde{b}_0^i = m_0k_0 \quad (16)$$

for all i . The repayment strategy gives optimal policy rules $k(t, \tilde{t})$, $\tilde{b}(t, \tilde{t})$, $h(t, \tilde{t})$, for any $t \geq 0$ and any realization of $\tilde{t} \geq 0$. If, for some t , the

optimal policies imply $\tilde{b}(t, t) > m(0)k(t, t)$, then the repayment strategy requires default at t if $t = \tilde{t}$. In this case, for any $t > \tilde{t}$ the policy rules are random variables, since they depend on realization of the stochastic moment τ (with arrival rate θ) that determines the time $\tilde{t} + \tau$ at which a household is readmitted to the financial market. The realization of τ is i.i.d. across households. Using the law of large numbers we derive, for pair any $(t, \tilde{t}) \geq 0$, the aggregate quantities K and H implied by the optimal individual policies,

$$\begin{aligned} K(t, \tilde{t}) &= E[k(t, \tilde{t})] \\ H(t, \tilde{t}) &= E[h(t, \tilde{t})] \end{aligned} \tag{17}$$

Define also

$$\begin{aligned} G(t, \tilde{t}) &= \frac{\dot{K}(t, \tilde{t})}{K(t, \tilde{t})} \\ G^H(t, \tilde{t}) &= \frac{\dot{H}(t, \tilde{t})}{H(t, \tilde{t})} \end{aligned} \tag{18}$$

We are now ready to explain how the arrival rate π_0 and the price function $w(\tau)$ are determined endogenously.

2.2 Aggregate Technology and General Equilibrium

Assume that the cost w is technologically related to the aggregate amount H of the composite input provided to the final good sector in the following way

$$w(H) = \begin{cases} w_0 & H \leq \bar{H} \\ w_0 + \gamma \left[\left(\frac{H}{\bar{H}} \right)^{1+\gamma} - 1 \right] & H > \bar{H} \end{cases} \tag{19}$$

with $\gamma > 0$ and for some constant $\bar{H} > 0$. This technology captures the following intuition. At the early stages of development of an economy, some inputs might be underutilized. As a consequence, larger and larger amounts of such inputs can be provided to the final good sector at a fairly constant marginal cost. As the economy develops, and more of the composite input H is used, its cost of production increases. The choice of a polynomial function for the cost w at levels $H > \bar{H}$ is done for analytical convenience. Notice that the curvature of $w(\cdot)$ is controlled by γ and that, at $H = \bar{H}$, the function w is not differentiable. It seems natural to consider the case where the cost w does not start to increase abruptly at $H = \bar{H}$. Therefore, I will typically consider small values for γ , which give a smoother function $w(\cdot)$. A second important aspect is that the aggregate technology is stochastic,

since the turning point \bar{H} is assumed to be drawn from a Pareto distribution with parameters³ $\eta \in (0, 1)$ and $H_{min} > 0$. Consequently, economic agents have a unconditional belief on the realization of \bar{H} given by

$$Prob(\bar{H} \leq H) = 1 - \left(\frac{H_{min}}{H}\right)^\eta$$

for $H \geq H_{min}$ ⁴. The reasons for the choice of a Pareto distribution will be clear in a moment. For now, I show with an example how the assumptions on the aggregate production technology allow us to capture the following three features for a growth process. First, there are two stages of growth, one characterized by constant marginal returns to capital, followed by a stage with strictly decreasing returns. Second, the timing of the transition from stage one to stage two is uncertain. Third, at any time during stage one, the probability of transitioning to stage two increases with the growth rate of the economy.

Suppose that, starting at time 0 from a level $H_0 = H_{min}$, the amount of H used into production grows at a constant rate $G_0^H > 0$. At a certain unknown time \tilde{t} the turning point \bar{H} will be reached. The time \tilde{t} is a random variable defined implicitly by

$$\bar{H} = H_0 e^{G_0^H \tilde{t}}$$

Suppose that, after \tilde{t} , the aggregate input H grows at rate $G^H(t, \tilde{t}) \geq 0$. Then the evolution of the cost $\tilde{w}(t, \tilde{t})$ satisfies

$$\dot{\tilde{w}}(t, \tilde{t}) = w_0 \quad t \leq \tilde{t} \quad (20)$$

$$\dot{\tilde{w}}(t, \tilde{t}) = (1 + \gamma)[\tilde{w}(t, \tilde{t}) - w_0 + \gamma]G^H(t, \tilde{t}) \quad t \geq \tilde{t} \quad (21)$$

The optimal choice of h is, for the household, a static problem. We can see that, for any $(t, \tilde{t}) \geq 0$, the optimal policy requires

$$(1 - \alpha)\tilde{A} \left[\frac{k(t, \tilde{t})}{h(t, \tilde{t})} \right]^\alpha = \tilde{w}(t, \tilde{t})$$

Constant returns to scale of the production function then gives

$$(1 - \alpha)\tilde{A} \left[\frac{K(t, \tilde{t})}{H(t, \tilde{t})} \right]^\alpha = \tilde{w}(t, \tilde{t}) \quad (22)$$

³For this parametrization, the Pareto distribution has no finite first and second moment. This is of no consequence in the model. Low values for η create a distribution with a fat tail, which create beliefs that place a lot of mass on large realizations of \bar{H} .

⁴As initial condition on the information set of the agents I assume that $w_0 = (1 - \alpha)\tilde{A} \left[\frac{K_0}{H_{min}} \right]^\alpha$.

which implies

$$G(t, \tilde{t}) - G^H(t, \tilde{t}) = \frac{1}{\alpha} \frac{\dot{\tilde{w}}(t, \tilde{t})}{\tilde{w}(t, \tilde{t})} \quad (23)$$

Notice that $G(t, \tilde{t}) = G^H(t, \tilde{t})$ for $t \leq \tilde{t}$. Equation (21) is rewritten as

$$\frac{\dot{\tilde{w}}(t, \tilde{t})}{\tilde{w}(t, \tilde{t})} = \frac{\alpha}{1 - \alpha} a(t, \tilde{t}) G(t, \tilde{t}) \quad (24)$$

where

$$a(t, \tilde{t}) = \frac{(1 - \alpha)(1 + \gamma)[\tilde{w}(t, \tilde{t}) - w_0 + \gamma]}{\alpha \tilde{w}(t, \tilde{t}) + (1 + \gamma)[\tilde{w}(t, \tilde{t}) - w_0 + \gamma]}$$

The marginal product of capital $A(t, \tilde{t})$ is defined by the function

$$A(t, \tilde{t}) \equiv \alpha \tilde{A} \left[\frac{K(t, \tilde{t})}{H(t, \tilde{t})} \right]^{\alpha-1} = \alpha \tilde{A} \left[\frac{(1 - \alpha) \tilde{A}}{\tilde{w}(t, \tilde{t})} \right]^{\frac{1-\alpha}{\alpha}} \quad (25)$$

The growth rate of A is then

$$\frac{\dot{A}(t, \tilde{t})}{A(t, \tilde{t})} = -(1 - \alpha)[G(t, \tilde{t}) - G^H(t, \tilde{t})] = -a(t, \tilde{t})G(t, \tilde{t}) \quad (26)$$

For $t \geq \tilde{t}$ we have $\tilde{w}(t, \tilde{t}) \geq w_0$ and then $a(t, \tilde{t}) \geq 0$. After the turning point the marginal product of capital decreases over time, as the economy keeps growing, and this is the main characteristic of the second stage of growth. For $t < \tilde{t}$ we have $\tilde{w}(t, \tilde{t}) = w_0$ and then $a(t, \tilde{t}) = 0$. The first stage of growth is characterized by a constant marginal product of capital A_0 ,

$$A_0 = \alpha \tilde{A} \left[\frac{(1 - \alpha) \tilde{A}}{w_0} \right]^{\frac{1-\alpha}{\alpha}}$$

To make the problem interesting I assume that $A_0 - \delta > \rho$. We now derive the arrival rate π_0 . Assume that, before the turning point, the growth rate of capital is constant and equal to G_0 , which is then also the growth rate of the input H . At time t before the turning point the total amount of input used into production is H_t . The probability that the turning point has been reached at time $\tilde{t} = t + \epsilon$ is

$$\pi_0 \epsilon = \text{Prob}\{\bar{H} \leq H_t e^{G_0 \epsilon} | H_t\} = 1 - e^{-\eta G_0 \epsilon} = \eta G_0 \epsilon + o(\epsilon) \quad (27)$$

As ϵ goes to zero we obtain that the instantaneous arrival rate of the turning point is $\pi_0 = \eta G_0$.

The evolution of the marginal product of capital that I have derived above is aimed at modelling the following intuition. When a poor country is at the early stage of a growth process it may not experience right-away the existence of decreasing marginal returns to capital. This can be the case, for instance, because there is an initially a large pool of underutilized resources. The unemployed input can be a mix of natural resources, labor (Lewis [1954]), slow human capital and technological accumulation (Chari and Hopenhayn [1991]). However, as more of the initially unemployed inputs are used into production, the marginal cost of their provision starts to increase, and this generates a reduction in the marginal product of capital. Another interpretation is that, at the turning point, the process of growth is slowed down because the economy hits a country-specific technological barrier (Parente and Prescott [1994]). The faster an economy grows, the quicker the economy will hit its barrier, whose *level* is nonetheless uncertain. Different countries can have different levels of the barrier. While decreasing marginal returns eventually appear and all the countries converge to a steady state, the economy that has drawn a higher level of \bar{H} will feature a higher steady state level of output. Seen from a different angle, this is indeed a model of *conditional convergence*, where the steady state level of output is uncertain.

The assumption that \bar{H} is distributed as a Pareto distribution has an important implication. In fact, at any time t , the (instantaneous) conditional probability of reaching the turning point depends only on the growth rate G_0 . The level of H_t does not matter per se, since it doesn't say anything about the conditional probability of hitting \bar{H} . The Pareto distribution has therefore a sort of "memoryless" property which allows the structure of uncertainty to be unchanged as the economy grows at a constant rate.

Definition 1. *An equilibrium is given by leverage bounds $\tilde{m}(t, \tilde{t})$, prices $\tilde{w}(t, \tilde{t})$, $\tilde{r}(t, \tilde{t})$, an arrival rate π_0 , initial conditions (16) and policies $k(t, \tilde{t})$, $\tilde{b}(t, \tilde{t})$, $h(t, \tilde{t})$ such that*

- i) The leverage bounds are sustainable: along the equilibrium path conditions (12), (13) hold and $\tilde{m}(t, \tilde{t}) = m(t - \tilde{t})$ for $t \geq \tilde{t}$.*
- ii) Optimality for the household: given prices and π_0 , the policies are obtained from the repayment strategy.*
- iii) Consistent cost function: $\tilde{w}(t, \tilde{t})$ satisfies (20) and (24), where $G(t, \tilde{t})$ is given by (18), and $\tilde{w}(t, \tilde{t}) = w(t - \tilde{t})$ for $t \geq \tilde{t}$.*
- iv) Consistent arrival rate: $G(t, \tilde{t}) = G_0$, $\forall(t, \tilde{t})$, with $t < \tilde{t}$, and $\pi_0 = \eta G_0$.*

v) *Fair pricing*: $\tilde{r}(t, \tilde{t})$ satisfies (14) and (15).

Notice that any equilibrium features a balanced growth path prior to the turning point, where all the variables grow at a rate G_0 . To conserve on notation, and whenever this does not create confusion, we use the subscript t to indicate a variable at time t without indicating the realization of the turning point \tilde{t} . Using the optimality condition in the choice of h we write the net output as

$$\tilde{A}k_t^\alpha h_t^{1-\alpha} - \tilde{w}_t h_t = A_t k_t$$

with A_t defined by (25). The next lemma provides two important features of any equilibrium. The first is that the model can be detrended, allowing us to study a stationary problem. The second regards the consequence of relaxing the constraint (10). The lemma shows that at any instant either the leverage bound is binding or the constraint (10) is binding. In this case, as the bounds D_0 and \bar{D} go to infinity, the leverage ratio tends to “jump” instantaneously (i.e. have an infinite growth in an infinitesimal span of time) to the leverage bound. By appropriately taking the limits we progressively approximate the solution to a problem where the state variables of the households are allowed to follow discontinuous paths and have jumps. I refer to this problem as the *model with jumps*. The Lemma also derives the balanced growth rate G_0 .

Lemma 1. Define $b = \frac{\tilde{b}}{k}$. In any equilibrium the following properties hold

a)

$$\begin{aligned} \bar{V}^d(k, \tau) &= \bar{V}^d(\tau) + \frac{\log k}{\rho} & V^d(k) &= V^d(1) + \frac{\log k}{\rho} \\ \bar{V}(k, \tilde{b}, \tau) &= \bar{V}(1, b, \tau) + \frac{\log k}{\rho} & V(k, \tilde{b}) &= V(1, b) + \frac{\log k}{\rho} \end{aligned} \quad (28)$$

b) Define $\eta^* = 0$ if $m_0 \leq m(0)$ and $\eta^* = \eta$ otherwise. Then

$$G_0 = \frac{A_0 - \delta - \rho}{1 - (1 - \eta^*)m_0} \quad (29)$$

c) At any time, either the leverage ratio is binding or (10) is binding. Moreover,

$$\begin{aligned} \lim_{\bar{D}(\tau) \rightarrow \infty} \bar{V}(1, b, \tau) &= \bar{V}(1, \bar{m}(\tau), \tau) + \frac{1}{\rho} \log \frac{1-b}{1-\bar{m}(\tau)} \\ \lim_{\bar{D}_0 \rightarrow \infty} V(1, b) &= V(1, m_0) + \frac{1}{\rho} \log \frac{1-b}{1-m_0} \end{aligned} \quad (30)$$

Proof. The proof of part *a)* of Lemma 1 is reported in Appendix A where I also show that at any time t either the leverage bound is binding or (10) is binding. I now derive the limiting jump process for $\bar{D}(\tau)$ that goes to infinity. A similar argument can be used for \bar{D}_0 and is left to the reader. Define $g_\tau = \frac{\dot{k}_\tau}{k_\tau}$. For simplicity define the bound \bar{D} as follows: $\bar{D}(\tau, b_\tau, g_\tau) = -\rho b_\tau + g_\tau b_\tau + D$ for some constant $D > 0$. Equation (10) is rewritten as

$$\dot{b}_\tau \leq D$$

Consider any time τ after the turning point and assume that $b_\tau < m(\tau)$. By continuity of m_t we have

$$m(\tau + \epsilon) = m(\tau) + u(\epsilon)$$

for some function $u(\epsilon)$ such that $u(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Choose D such that

$$D = \frac{m(\tau) + u(\epsilon) - b_\tau}{\epsilon}$$

It is easy to check that

$$b_{\tau+\epsilon} = b_\tau + D\epsilon = m(\tau + \epsilon)$$

It takes exactly a span of time ϵ for the leverage ratio to reach the leverage bound starting from a value $b_\tau < m(\tau)$.

For any $\hat{\tau}$ such that $\tau \leq \hat{\tau} < \tau + \epsilon$ the first order condition with respect to $g_{\hat{\tau}}$ implies

$$c_{\hat{\tau}} = \rho(1 - b_{\hat{\tau}}) = A(\hat{\tau}) - \delta - g_{\hat{\tau}} + d_{\hat{\tau}} = A(\hat{\tau}) - \delta - (1 - b_{\hat{\tau}})g_{\hat{\tau}} + D - \rho b_{\hat{\tau}}$$

which gives

$$g_{\hat{\tau}} = \frac{A(\hat{\tau}) - \delta - \rho + D}{(1 - b_{\hat{\tau}})} = \frac{A(\hat{\tau}) - \delta - \rho + D}{1 - b_\tau - \bar{D}(\hat{\tau} - \tau)}$$

Integrating in $\hat{\tau}$ from τ to $\tau + \epsilon$ we obtain

$$\begin{aligned} \log \frac{k_{\tau+\epsilon}}{k_\tau} &= -\frac{A(\tau) - \delta - \rho + D}{D} \log \frac{1 - b_\tau - D\epsilon}{1 - b_\tau} \\ &= -\frac{A(\tau) - \delta - \rho}{m(\tau) + u(\epsilon)} \epsilon \log \frac{1 - m(\tau + \epsilon)}{1 - b_\tau} - \log \frac{1 - m(\tau + \epsilon)}{1 - b_\tau} \end{aligned}$$

Taking limits for $\epsilon \rightarrow 0$ we finally have

$$\log \frac{k_{\tau+\epsilon}}{k_\tau} \rightarrow \log \frac{1 - b_\tau}{1 - m(\tau)} \quad (31)$$

For a given ϵ small (and thus for a given large D), and the properties in part *a*) we can write

$$\begin{aligned}\bar{V}(k, \tilde{b}, \tau) &= \bar{V}(1, b, \tau) + \frac{\log k_\tau}{\rho} \\ &= (\log c + \log k_\tau)\epsilon + (1 - \rho\epsilon) \left[\bar{V}(1, m(\tau + \epsilon), \tau + \epsilon) + \frac{\log k_{\tau+\epsilon}}{\rho} \right] + o(\epsilon)\end{aligned}$$

or

$$\bar{V}(1, b, \tau) = \log(c)\epsilon + (1 - \rho\epsilon) \left[\bar{V}(1, m(\tau + \epsilon), \tau + \epsilon) + \frac{1}{\rho} \log \frac{k_{\tau+\epsilon}}{k_\tau} \right] + o(\epsilon)$$

For $\epsilon \rightarrow 0$ we finally obtain

$$\bar{V}(1, b, \tau) \rightarrow \bar{V}(1, m(\tau), \tau) + \frac{1}{\rho} \log \frac{1 - b_\tau}{1 - m(\tau)}$$

□

Denote with $V(b)$, V^d , $\bar{V}(b, \tau)$, $\bar{V}^d(\tau)$ and $\bar{V}^0(b)$ respectively the detrended value functions $V(1, b)$, $V^d(1)$, $\bar{V}(1, b, \tau)$, $\bar{V}^d(1, \tau)$ and $\bar{V}^0(1, b)$. We only consider solutions to the household's problem for the model with jumps, where value and policies functions are obtained as a solution for the limit in part *c*) of Lemma 1. Since, by part *c*), the leverage bound is always binding we have

$$\begin{aligned}\rho \bar{V}(m, \tau) &= \max_g \log c + \bar{V}'_b(m, \tau) \dot{m} + \bar{V}'_\tau(m, \tau) + \frac{g}{\rho} \\ \text{s.t. } c &= A(\tau) - \delta + \dot{m} - g - m(\rho - g)\end{aligned}\tag{32}$$

$$\begin{aligned}(\rho + \pi_0)V(m_0) &= \max_{g_0} \log c + \pi_0 V^0(m_0) + \frac{g_0}{\rho} \\ \text{s.t. } c &= A_0 - \delta = g_0 - m_0(r_0 - g_0)\end{aligned}$$

By part *c*), the leverage ratio jumps immediately from zero to the leverage bound when the household is readmitted to the financial market after

default, then

$$\begin{aligned} (\rho + \theta)\bar{V}^d(\tau) &= \max_{g^d} \log c^d + \theta \left[\bar{V}(m(\tau)) - \frac{\log(1 - m(\tau))}{\rho} \right] + \bar{V}'^d_\tau(\tau) + \frac{g^d}{\rho} \\ \text{s.t. } c^d &= \xi A(\tau) - \delta - g^d \end{aligned}$$

$$\begin{aligned} (\rho + \theta + \pi_0)V^d &= \max_{g_0^d} \log c^d + \theta \left[V(m_0) - \frac{\log(1 - m_0)}{\rho} \right] + \pi_0 \bar{V}^d(0) + \frac{g_0^d}{\rho} \\ \text{s.t. } c^d &= \xi A_0 - \delta - g_0^d \end{aligned} \tag{33}$$

Again, part *c*) implies that, if the household has a capital k_t at the moment of the readmission to the financial market, her capital jumps immediately to $\frac{1}{1-m_t}k_t$. The quantity $\frac{1}{1-m_t}$ is then a capital multiplier, which takes place thanks to the leverage effect of international borrowing.

The first order conditions at any time $\tilde{t} + \tau$ after the turning point are

$$c = \rho(1 - \bar{m}) \tag{34}$$

$$c^d = \rho \tag{35}$$

$$g = \frac{A - \delta - \rho + \dot{m}}{1 - m} \tag{36}$$

$$g^d = \xi A - \delta - \rho \tag{37}$$

Before moving to the next section, I make sure that, during default, the net marginal product of capital never falls below the risk free rate ρ and assume the following function for the output costs

$$\xi(\tau) = \max \left\{ \xi, \frac{\rho + \delta}{A(\tau)} \right\} \quad \tau \geq 0 \tag{38}$$

for some constant $\xi \in (0, 1)$ such that $\xi A_0 > \delta + \rho$. Notice that (38) implies that $\xi(\tau)A - \delta \geq \rho$, i.e. even in presence of output costs the net marginal product of capital never falls below the international risk-free rate ρ . This assumption is made for analytical convenience, since it gives

$$g^d_\tau = \max\{\xi A(\tau) - \delta - \rho, 0\}$$

Notice, finally, that the assumption that the punishment for default is autarky boils down to assuming that the household cannot borrow during the punishment, but could lend to international agents at the risk free rate ρ . However, this is never optimal, since (38) implies that the return on capital $\xi A(\tau) - \delta$ is always bigger than the international risk free rate.

2.3 The endogenous leverage bounds

There are many leverage bound functions m_0 and $m(\tau)$ which satisfy (12) and (13) as, for instance, the autarky functions $m_0 = m(\tau) = 0$. In this section I construct particular function that define what I call “endogenous leverage bounds”. I first derive bounds after the turning point and I will say that $m(\tau)$ defines *endogenous leverage limits* if and only if, in equilibrium,

$$\bar{V}(m(\tau), \tau) = \bar{V}^d(\tau) \quad \forall \tau \geq 0 \quad (39)$$

The endogenous leverage bounds have the property that they make an household indifferent, at any state, between repaying and defaulting on the debt. Given such indifference, I assume that the household always chooses the repayment strategy.

Condition (39) implies the following

$$\frac{d}{dt}\bar{V}(m(\tau), \tau) = \frac{d}{dt}\bar{V}^d(\tau) \quad \forall \tau \geq 0$$

Rewriting the total differential with respect to time we have

$$\bar{V}'_b \dot{m} + \bar{V}'_\tau - \rho \bar{V}(m, \tau) = \bar{V}'_\tau - \rho \bar{V}^d(\tau) \quad \forall \tau \geq 0 \quad (40)$$

Plugging (40) into (33) and using (34)-(35), yields an equation that defines, at any state, the endogenous leverage bounds

$$\rho \log \frac{c_\tau}{c_\tau^d} - \theta \log \frac{1}{1 - m(\tau)} + (g_\tau - g_\tau^d) = 0 \quad (41)$$

The incentives that must be provided to the household to make her indifferent between following the repayment strategy or the default strategy are summarized by the three terms in the sum (41). The inspection of (34)-(35) shows that the first term is always negative. In fact, consumption under the repayment strategy is smaller than that under the default strategy, since the possibility of leveraging investment through international borrowing induces the household to choose higher investment rates, at the expenses of current consumption. This is the *consumption effect*. The second term is also negative, and is the *debt forgiveness effect*, or the wealth gain that the household would make if she were to default and be readmitted instantaneously, with zero debt, to the financial market, an event with arrival rate θ . When the leverage bounds are endogenous the third term, the *excess growth effect*, is positive and exactly compensates the first two. The growth rate of capital under repayment must be larger than the growth in default and this extra

growth represents a long run gain in utility stemming from an increase in the wealth of the household. Using (36)-(37) we find

$$g_\tau - g_\tau^d = \frac{m(\tau)}{1 - m(\tau)} [A(\tau) - \delta - \rho] + \frac{\dot{m}(\tau)}{1 - m(\tau)} + [1 - \xi(\tau)]A(\tau)$$

Again we divide the excess growth $g - g^d$ into three terms. The first is the part of excess growth coming from the possibility of leveraging investment through international borrowing, captured by the capital multiplier. The second is the effect on growth of a variation in the leverage bound, which is amplified by the capital multiplier. If $\dot{m} < 0$ excess growth is reduced, because *de-leverging* the economy determines a reduction of resources available for investment. The third term is the part of excess growth enjoyed by a household following the repayment strategy due to the absence of the output costs arising from the choice of defaulting.

Denote with $\phi(\tau)$ the fraction of the aggregate capital that at time $\tilde{t} + \tau$ is allocated to households excluded from the financial market because of default at time \tilde{t} . Clearly, if $m(0) < m_0$ then the repayment strategy requires that all the households default simultaneously at \tilde{t} and then $\phi(0) = 1$. Otherwise, if there is no default at the turning point, $\phi(0) = 0$. At any τ the capital of all the households with access to the financial market grows at the same rate g_τ , while the capital of all the households in default grow at rate g_τ^d . Moreover, at a rate θ , households in default regain access to the financial and their capital jumps immediately proportionally to the capital multiplier $\frac{1}{1 - m(\tau)}$. In Appendix B I show that the growth rate G_t of aggregate capital at any time t is

$$G_\tau = (1 - \phi_\tau)g_\tau + \phi_\tau g_\tau^d + \phi_\tau \theta \frac{m(\tau)}{1 - m(\tau)} \quad (42)$$

As shown in Appendix B, the evolution of the share ϕ of capital allocated to households in default is

$$\dot{\phi}_\tau = -\phi_\tau(\theta + G_\tau - g_\tau^d) \quad (43)$$

After substituting for c , c^d , g and g^d equation (41) can be written as a differential equation in explicit form⁵

$$\dot{m} = f_2(m, A) \quad (44)$$

for some function $f_2(m, A)$. With the appropriate substitutions for G , g and g^d , (26), (43), (44) define a system of three differential equations in explicit

⁵The function f_2 is reported explicitly in the proof of Lemma 2.

form in the variables m , A and ϕ . A solution to this system gives, as a function of time, an evolution for the endogenous borrowing limits after the turning point. The initial conditions are given by

$$A(0) = A_0, \quad \phi(0) \in \{0, 1\}$$

Lemma 2 states the existence and characterize any solution to the system (26), (43), (44). While all the result that I derive hold in a general setting, to make the exposition simpler I assume that

$$\{m | f_2(m, A_0) = 0\} \neq \emptyset \quad (45)$$

Condition (45) is basically a requirement that A_0 is not too large⁶. The equation $f_2(m, A_0) = 0$ typically has two solutions and I call m_2 the smaller of the two

$$m_2 = \min\{m | f_2(m, A_0) = 0\} \quad (46)$$

Lemma 2. *For any initial condition $\phi(0) \in [0, 1]$ and $A(0) = A_0$, the system (26), (43), (44) has a solution $m(\tau)$, $A(\tau)$, $\phi(\tau)$. Moreover,*

$$m(0) = m_2 - \epsilon_\gamma$$

$0 \leq \epsilon_\gamma \leq \nu(\gamma)$ for some continuous function $\nu(\cdot)$ such that $\nu(0) = 0$. The functions $m(\tau)$ and $A(\tau)$ are decreasing, with

$$\lim_{t \rightarrow \infty} m(\tau) = 0, \quad \lim_{t \rightarrow \infty} A(\tau) = \rho + \delta$$

If $\phi(0) = 0$ the solution is unique.

The proof is reported in Appendix A. We can gain intuition on the results of Lemma 2 by looking at Figure 4, which plots the phase diagram $f_2(m, A)$ for different values of A . For any τ such that $m(\tau) > 0$ equation (41) requires that $g_\tau > g_\tau^d \geq 0$. Then, by (42) and (26) we have $\dot{A}_t < 0$, which means that after the turning point the marginal product of capital decreases over time. Consequently, *ceteris paribus*, the excess growth $g_\tau - g_\tau^d$ tends to decrease, which by (41) implies that $m(\tau)$ should decrease. However, the required deleveraging, $\dot{m}(\tau) < 0$, would tend to further reduce the excess growth and the incentives for the household to follow the repayment strategy. If the household reaches the turning point with a high leverage ratio $m(0) > m_2$ the necessary deleveraging process is not sustainable. In fact, as the marginal

⁶ $A(\tau)$ is a decreasing function and, eventually, condition (45) holds for some time τ sufficiently large.

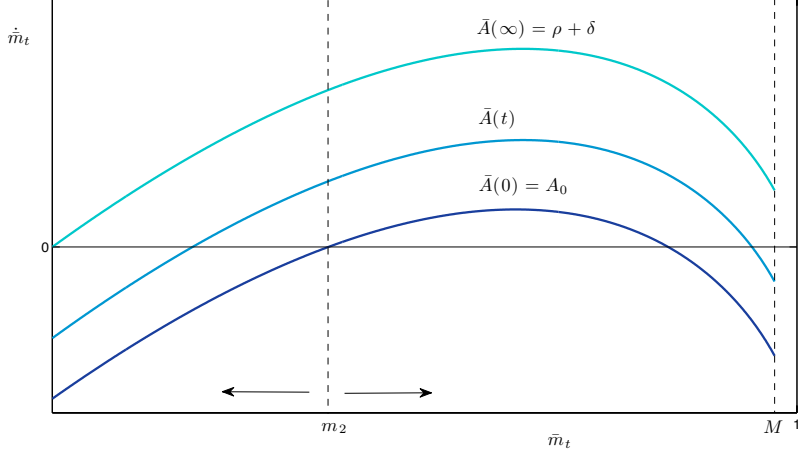


Figure 4: The endogenous leverage bound after \tilde{t} .

product of capital decreases, the only way to provide incentives for the repayment strategy would be to increase the leverage ratio, $\dot{m}(\tau) > 0$, which eventually hits the bound M .

Once we have obtained a solution $m(0)$ we can work backwards to find the endogenous leverage bound prior to the turning point. It is natural to define the endogenous leverage bound m_0 as

$$m_0 = \max\{m | V(m) \geq V^d\} \quad (47)$$

If the endogenous leverage bounds are such that $m_0 \leq m(0)$ then there is no default at the turning point. Instead, if $m_0 > m(0)$ then, by construction, the repayment strategy requires default at the transition point between the first and the second stage of growth. We can now see that such a construction for the repayment strategy wasn't arbitrary. In fact, recall that by Lemma 2 $m(0)$ is arbitrarily close m_2 . Then, if $m(0) > m_0$ we can say that, for γ small, $m_0 > m_2$. An household then reaches the turning point with a high leverage but, by the discussion in Figure 4, there is no credible path of deleveraging starting from values higher than m_2 , and default thus takes place.

Similarly to (41) we find

$$\rho(\rho + \theta + \pi_0)[V(m_0) - V^d] = (\rho + \theta) \log(1 - m_0) + g_0 - g_0^d$$

where g_0^d is given by (37) and g_0 equals (29). Using the fact that $m(0) < m_2$ we can easily show that $V(m(0)) - V^d \geq 0$ and then $m_0 = m(0)$ is always

sustainable. In particular, as γ goes to zero, the inequality (47) holds with equality

$$\lim_{\gamma \rightarrow 0} V(m(0)) - V^d = f_2(m_2, A_0) = 0$$

The possible shapes of $\bar{V}(m) - V^d$ are shown in Figure 5. As already mentioned, a leverage bound equal to $m_0 = m(0)$ is sustainable and the difference between $V(m(0))$ and V^d is close to zero. A discontinuity takes place at $m(0)$ because, as we increase m , the interest rate changes discontinuously from ρ to $\rho + \pi_0$. Levels of m close to $m(0)$ are then not sustainable, since the consumption and debt forgiveness effects dominate over the excess growth effect. If we further increase m to intermediate values we could have that, as for the solid line in Figure 5, the high leveraging boosts the growth effect, which then dominates and makes the leverage bound sustainable again. Finally, if we keep increasing the leverage m , the excess growth effect loses in importance because its rise is dampened by the corresponding rise in the risk premium $\pi_0 = \eta G_0$. The resulting higher interest rate works in the direction of reducing both the aggregate growth G_0 and the excess growth effect. For the case of the solid line in Figure 5 we then obtain an endogenous leverage ratio $m_0 > m(0)$ and the repayment strategy requires default at the turning point. There are, however, paths of growth where no default takes place. In fact, if A_0 is sufficiently low, the function $V(m) - V^d$ follows the shape of the dashed line. In this case, due to the low marginal product of capital, the excess growth effect never dominates and the endogenous borrowing limit is $m_0 = m(0)$.

For $m > m(0)$ the interest rate in the first stage of growth is $\rho + \eta G_0$, we find that

$$\begin{aligned} f_1(m) &\equiv \rho(\rho + \pi_0)[V(m) - V^d] \\ &= (\rho + \theta) \log(1 - m) + \frac{(1 - \eta)m}{1 - (1 - \eta)m} (A_0 - \delta - \rho) + (1 - \xi)A_0 \end{aligned} \quad (48)$$

The proof of the next proposition is reported in Appendix A.

Proposition 1. *Define an economy by a vector $\mathcal{E} \equiv (A_0, \frac{1}{\theta}, 1 - \xi)$ and assume that γ is small. Define m_2 as in (46), $f_1(m)$ as (48) and M_1 by*

$$M_1 = \{m | m \geq m_2, f_1(m) = 0\}$$

If $M_1 = \emptyset$ then $m_0 = m(0)$ and the equilibrium is unique and with no default. If $M_1 \neq \emptyset$ then $m_0 = \max\{M_1\} > m(0)$, any equilibrium has default at the turning point, and $V(m_0) = V^d$.

Consider two economies such that $\mathcal{E}' \gg \mathcal{E}$. Then, if default occurs in \mathcal{E} it also occurs in \mathcal{E}' .

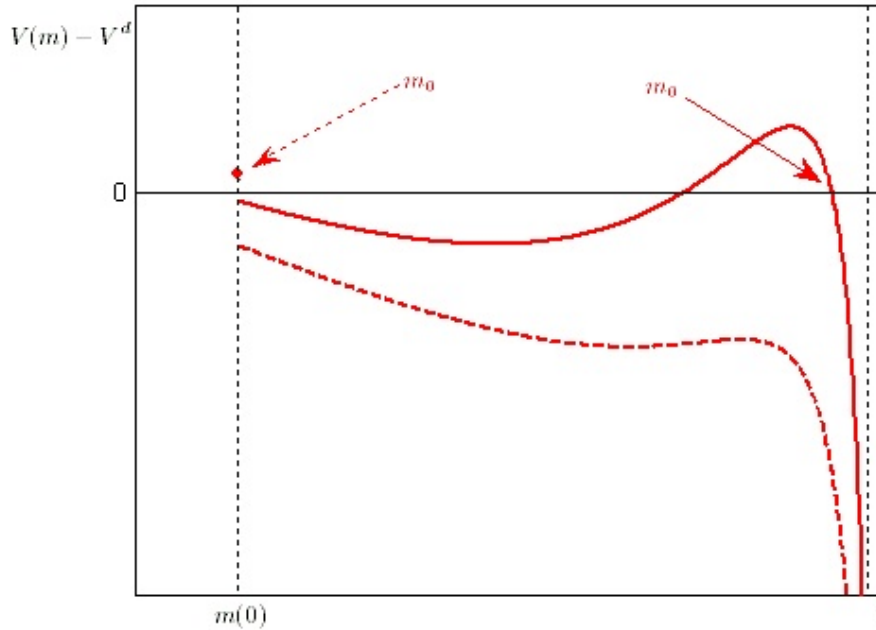


Figure 5: The endogenous leverage bound before \tilde{t} .

In this model, there are only two possibilities for a path of growth. An economy can accumulate, during its first stage of development, a large stock of debt which fuels a high growth rate. Default takes place at the turning point, when decreasing marginal returns on capital start to appear. After the economy recovers from the financial crisis, and the recession due to output costs, the growth rate is lower because it is chocked by a low ability to finance investment through international borrowing. In the second stage of growth, in fact, the country experiences a sudden stop in the capital inflow, which resembles a permanent credit crunch following a credit boom. This situation is depicted by the path of the leverage bound shown in Figure 6. The second possibility is that the accumulation of debt during the first stage of growth is moderate. The growth rate is smaller and the economy transitions to the stage characterized by decreasing marginal returns without defaulting. The two different paths for the GDP growth rate are shown in Figure 7. Similar patterns are followed by the investment rate in Figure 8. Notice how the economy following an equilibrium with defaults looks more

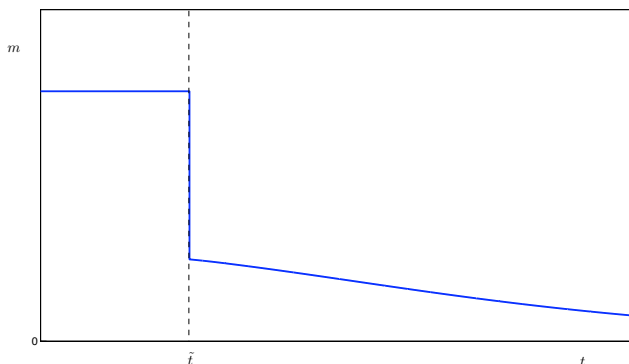


Figure 6: Path of the endogenous leverage bound in an equilibrium with default.

volatile than an economy on an equilibrium with no default. The fact that an economy ends up following one path or the other is decided by the vector $\mathcal{E} \equiv (A_0, \frac{1}{\bar{\theta}}, 1 - \xi)$. If the initial marginal product of capital A_0 is high the excess growth effect is strong. This expands the endogenous leverage ratio and the economy follows the path featuring initial high debt-high growth-high risk and consequent default. This might be, for instance, the growth path of an economy which is initially very underdeveloped. On the contrary, a more developed economy featuring a lower marginal product of capital will follow a smoother path of growth, with smaller growth rates which do not end up in a financial crisis. Similar comments can be made for the punishment parameters. Larger output costs $1 - \xi$ increase the excess growth effect, while longer exclusion $\frac{1}{\bar{\theta}}$ from the financial market decreases the debt forgiveness effect. Stronger punishment increases debt sustainability and tends to set the economy on an initially higher path of growth. However, the tougher the punishment, the harder the consequences of the default that takes place at the turning point, both in terms of output costs and in terms of longer autarky. Stronger punishments tend then to make more volatile the development process of the country.

The next section is devoted to the dynamics of the current account. I show that, when we allow for bubble formation, the model is capable of generating sudden and permanent reversals in the current account, following a financial crisis. I show also that the presence of a bubble in the current account allows for debt sustainability also in the context of Bulow and Rogoff (1989).

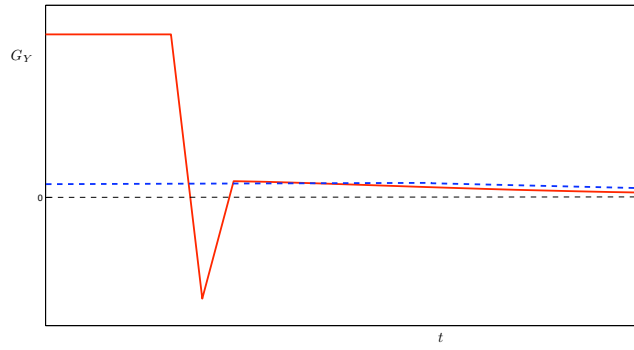


Figure 7: GDP growth: equilibrium with default (solid) and without default (dashed).

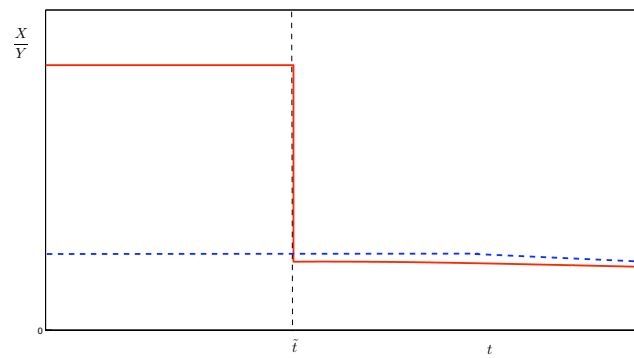


Figure 8: Investment rate: equilibrium with default (solid) and without default (dashed).

2.4 Bubbles, Debt and the Current Account

In their influential paper, Bulow and Rogoff (1989) showed that no contingent debt contract can be enforced between a lender and a borrower whenever the only punishment for a defaulting borrower is the permanent exclusion from future borrowing (but not from lending). I show that in my model the Bulow-Rogoff (from now BR) result is strictly connected to a transversality condition that rules out bubbles.

The formulation of BR is easily replicated here by setting $\theta = 0$ and $\xi = 1$. It is straightforward to see that the assumption that the household can invest abroad after defaulting is inconsequential, since at each point in time $A_t - \delta \geq \rho$.

We find that

$$m_2 = 0$$

Since, by Lemma (2), an endogenous leverage bound $m(\tau)$ must satisfy $0 \leq m(\tau) \leq m_2$, we conclude that $m(\tau) = 0$ for all $\tau \geq 0$ is the unique endogenous leverage bound after the turning point, a result consistent with BR. However, before the turning point, there can be an endogenous leverage bound that sustains a strictly positive amount of debt. By Proposition (1) we know that this is possible if and only if

$$\max\{M_1\} = \max\{m|f_1(m)\} > 0$$

It is easy to see that the absence of output costs implies $f_1(0) = 0$. Moreover,

$$\frac{df_1(0)}{dm} = (1 - \eta)(A_0 - \delta - \rho) - \rho$$

Therefore, if A_0 is sufficiently large, we have $\frac{df_1(0)}{dm} > 0$ which implies that there exists a solution $m_0 > 0$ to $f(m_0) = 0$. Hence, if the marginal product of capital is sufficiently high, the unique equilibrium with endogenous leverage bounds features a balanced growth path prior to the turning point with strictly positive debt. The debt contract is contingent on the stochastic moment where the economy reaches the turning point and can be expressed as follows: repay the debt, which grows at an instantaneous rate G_0 , as long as the turning point is reached, then default and enter autarky for ever. Why does the BR result seem to fail here?

Recall that $r_0 = \rho + \eta G_0$ and G_0 is given by (29). We can write $f_1(\cdot)$ as

$$f_1(m_0) = \rho[\log(1 - m_0) + m_0] - m_0(r_0 - G_0)$$

It is easy to see that

$$f_1(m_0) > 0 \Rightarrow (r_0 - G_0) < 0 \quad (49)$$

The balanced growth path prior to default is non autarkic and strictly positive debt can be sustained only if the effective interest rate $r_0 - G_0$ is negative. In other words, if the marginal product of capital A_0 is high enough, the aggregate capital and the aggregate debt grow at a rate bigger than the interest rate. In this situation the household continuously rolls over her current debt obligations by issuing new debt and has no incentive to default.

To follow the notation of BR define $D^0(t)$ as the time zero *value of the payments* $-\tilde{d}_\tau$ up to time t that international investors receive from the small open economy. After the turning point the economy is in autarky, then $\tilde{d}_\tau = 0$. Instead, prior to the turning point, $-\tilde{d}_\tau = -d_0 k_t = (r_0 - G_0)m_0 k_0 e^{G_0 \tau}$. The arrival rate of the turning point is $\pi_0 = \eta G_0 = r_0 - \rho$, then

$$D^0(t) = E \left[- \int_0^\infty e^{-\rho \tau} \tilde{d}_\tau d\tau \right] = (r_0 - G_0) \tilde{b}_0 \int_0^t e^{-(r_0 - G_0) \tau} d\tau < 0$$

The net present value at time zero of the households payments is then $D^0(\infty) = -\infty$. Any equilibrium where the present values of the payments to the lenders is negative is ruled out by BR. Debt contracts having a negative net present value might be considered non optimal for the lenders. However, this debt contract with negative net present value is of a particular sort. Recall that rationality for the lenders is here imposed in part *v*) of Definition 1 simply by requiring a fair pricing of the debt asset. A possible source of non optimality in the transversality condition that usually rules out bubble equilibria in asset pricing. In fact, define $p^0(t)$ the time zero price of an Arrow security that pays one unit of consumption good at time t if the turning point \tilde{t} has not been reached at t , and zero otherwise. The arrival rate of the turning point \tilde{t} is π^0 and fair pricing requires that

$$p^0(t) = e^{-(\rho + \pi_0)t} = e^{-r_0 t}$$

The equilibrium face value \tilde{b}_t of an household's debt at time t equals $\tilde{b}_0 e^{G_0 t}$ if the turning point has not been reached at t , and zero otherwise. The price at time zero of the *face value* of the equilibrium household's debt at time t is then $P^0(t) = p^0(t) \tilde{b}_t > 0$. We have

$$\lim_{t \rightarrow \infty} P^0(t) = \lim_{t \rightarrow \infty} e^{-(r_0 - G_0)t} \tilde{b}_0 = +\infty$$

The value $P^0(t)$ of the wealth invested by the international lenders into the economy goes to infinity as t increases. The usual transversality condition,

which requires that such value goes to zero, does not hold. Moreover, since $P^0(t) > D^0(t)$ for all t , the price P^0 of the debt asset is always greater than its “fundamental” D^0 . Hence, I name this situation a *bubble equilibrium*. Without a more precise specification of the problem solved by the international lenders, the simple requirement of a fair pricing is not enough to rule out bubble equilibria. Indeed, equilibria with rational bubbles can be obtained in a number of ways (see for instance Weil [1989], Jovanovic [2007], Hellwig and Lorenzoni [2007]). For the existence of an equilibrium with rational bubbles it is generally necessary that the bubble asset has a bounded supply (Jovanovic [2007]). Debt does not seem to fall in such a category of assets. Interestingly, in our model, the bound on the amount of debt that the country can issue is endogenously obtained because incentives to default don’t allow the leverage ratio to be greater than the endogenous leverage bound m_0 . At the same time, the very existence of a bubble provides incentives for the households not to default on the debt before the turning point, that we can consider as the moment where the bubble bursts. With this respect, our model resembles Hellwig and Lorenzoni (2007).

The discussion on the bubble equilibria introduces us to the problem of assessing whether the model can qualitatively account for the type of dynamics of the current account that we as seen in Figure 2. The answer to this question is affirmative, as I show in the remaining of this section. The question is addressed in a cleaner way if we make a straightforward extension of the model and we allow for partial default in the form of debt renegotiation. Assume that, at the turning point, an exogenous fraction $1 - \phi > 0$ of households have the option of choosing between complete default and renegotiating their current total debt $\tilde{b}_{\bar{t}} = m_0 k_{\bar{t}}$ to a new level $\bar{m} k_{\bar{t}}$. Renegotiation has the advantage of sparing the household the punishment inflicted to agents that default on their entire debt. If we assume that the renegotiation process extracts all the surplus from the borrowers in favor of international lenders, the level of the renegotiated debt is exactly $\bar{m} k_{\bar{t}} = m(0) k_{\bar{t}}$. In fact, by construction of the endogenous leverage bounds, $\bar{V}(m(0)) = V^d$, and the household is exactly indifferent between accepting and rejecting the renegotiation offer. The endogenous leverage bounds $m(\tau)$ solve the usual system of differential equation with initial condition⁷ $\phi(0) = \phi$. Also, for simplicity, assume that $\theta = 0$ so that the fraction ϕ of households that default disappear forever from any interaction with the financial market. On the contrary, households that renegotiated the debt, keep interacting with the international market.

⁷Lemma 2 proves existence of a solution for any $\phi(0) \in [0, 1]$.

The equilibrium interest rate r_0 takes into account that default is only partial and that the fact that international lenders are able to recover a fraction Δ of their investment,

$$\Delta = (1 - \phi) \frac{m(0)}{m_0}$$

Therefore,

$$r_0 = \rho + \pi_0(1 - \Delta) = \rho + \eta G_0(1 - \Delta) = \rho + \eta' G_0$$

where $\eta'(m_0) = \eta(1 - \Delta)$ is now a parameter that changes with m_0 , for a given $m(0)$ (which is independent on m_0). The endogenous leverage bound m_0 is found as before as the largest solution to $f_1(m) = 0$, where the function f_1 is now constructed with an endogenous parameter η' . This creates no major departure from the characterization of the equilibrium presented above for the case of complete default.

The dynamics of the current account of the country mirrors that of the households with access to the financial market. Prior to default, the (de-trended) net exports are

$$-d_0 = m_0(r_0 - G_0)$$

As we have seen above, it can be the case that $G_0 > r_0$, the country is a net importer and experiences a capital inflow. At any time $\tilde{t} + \tau$ after the turning point, net exports $-d_\tau$ are

$$-d_\tau = m(\tau)(\rho - g_\tau) - \dot{m}(\tau) > 0$$

with g_τ given by (36). If the sudden contraction in the leverage bound after the tuning point (Figure 6) is sufficiently sharp, the reduction in the growth rate can be strong enough that $g_\tau < \rho$ for all $\tau \geq 0$. Moreover, $\dot{m}(\tau) \leq 0$ since the economy is progressively deleveraged, which also contributes to the reduction in the growth rate. This situation is shown in Figure 9. Before the turning point the sustainable leverage bound can be very large, the growth rate of the economy is bigger than the interest rate and the country is in a credit boom. At the turning point the leverage bound falls sharply and the country experiences a credit crunch. Some borrowers can devalue (renegotiate) their debt, but the growth rate of the economy is very tightly constrained by the credit crunch. The lower growth rate coupled with the ongoing deleveraging process transforms permanently the country from a net importer into a net exporter.

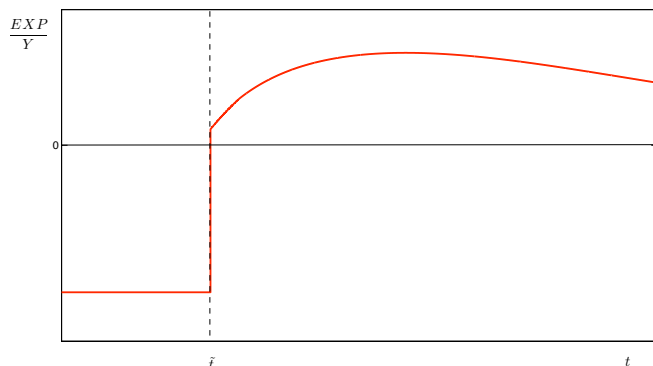


Figure 9: Net exports detrended by GDP for an equilibrium with default and debt renegotiation.

3 Conclusions

I constructed a model that links together, along a path of development of a small open economy, the growth rate of output, the direction of capital flows and the possibility of a financial crises. Households of the small economy borrow from international investors to finance the capital accumulation which drives the economic development. Default on the international debt is possible, which determines the existence of endogenous borrowing limits. The borrowing limits interact with the growth process giving rises to interesting dynamics whenever we introduce a simple element of uncertainty. Marginal returns to capital are initially constant and they start to decrease when the economy has reached an exogenously given level of development that I call “turning point”. Agents have only a probabilistic knowledge of the level of the turning point, and the speed at which it is reached depends on the growth rate of the economy, which in turn is determined by how tight are the endogenous borrowing constraints faced by the country. Two paths of development are possible. In one case, the borrowing limits are loose, the economy grows very rapidly until the turning point is reached. Following the initial credit expansion the economy suddenly faces a credit crunch which forces the households to default. The economy eventually recovers from the distressed periods, but the tightness of the borrowing constraints permanently reduces the growth of output and the investment rate. If the credit crunch is sufficiently severe the balance of payment is permanently reversed. This path is followed by economies with initial large marginal product of

capital or that face strong punishments for default. The second possibility is that the endogenous borrowing limits are tight from the very beginning. This produces a slower but more stable growth, since no default takes place at the turning point. I also show that there is a strict connection between debt sustainability and the existence of equilibria featuring the presence of bubbles.

4 References

Arellano C. (2007), *Default Risk and Income Fluctuations in Emerging Economies*, American Economic Review (forthcoming).

Barro R., Mankiw G., Sala-I-Martin X. (1995), *Capital Mobility in Neoclassical Models of Growth*, American Economic Review 1, 103-115.

Bulow, J., K. Rogoff (1989), *Sovereign Debt: Is to Forgive to Forget?*, American Economic Review 79 , 43-50.

Chari V.V., H. Hopenhayn (1991) , *Vintage Human Capital, Growth, and the Diffusion of New Technology*, Journal of Political Economy 99, 1142-65.

Franklin J., (1954), *On the Existence of Solutions of Systems of Functional Differential Equations*, Proceedings of the American Mathematical Society, Vol. 5-3: 363-369.

Guimaraes B (2008), *Optimal external debt and default*, Discussion Paper, Centre for Economic Performance, LSE.

Hellwig C., G. Lorenzoni (2007), *Bubbles and Self Reinforcing Debt*.

Jovanovic B. (2007), *Bubbles in Prices of Exhaustible Resource*, NBER.

Kehoe T. J., D.K. Levine (2000), *Liquidity Constrained Markets versus Debt Constrained Markets*.

Kehoe P., F. Perri (2002), *Business Cycles with Endogenous Incomplete Markets*, Econometrica 70, 907-928.

Lee, J.W., C. Rhee (2000), *Macroeconomic Impacts of the Korean Financial Crisis: Comparison with the Cross-country Patterns*, Rochester Center for Economic Research, working Paper No. 471.

Lewis W.A. (1954), *Economic Development with Unlimited Supplies of Labour*, The Manchester School, Vol. 22, pp. 139-191.

Lucas, R. (1990), *Why doesn't Capital Flow from Rich to Poor Countries?*, American Economic Review 80: 92-96.

Nehru V., A. Dhareshwar (1993), *A New Database on Physical Capital*, World Bank.

Parente S. L., E. Prescott (1994), *Barriers to technology adoption and growth*, Journal of Political Economy 102, 298-321.

Ranciere R., Tornell A, F. Westermann (2005), *Systemic Crises and Growth*, CESIFO Working Paper No. 1451.

Solow R. (1954), *A Contribution to the Theory of Economic Growth*, Quarterly Journal of Economics, 65-94.

Weil P. (1989), *Overlapping Families of Infinitely Lived Agents*, Journal of Public Economics 38, 183-198.

5 Appendix A

Proof of Lemma (1)

Proof. For part a) I prove the property for $\bar{V}(k_t, \tilde{b}_t, t)$, while the proof for the other value functions is left to the reader. At any time $\tilde{t} + t$ after the turning point \tilde{t} define $\hat{V}(b_t, t)$ the function that solves

$$\rho \hat{V}(b_t, t) = \max_{g_t, d_t} u(c_t) + \hat{V}'_b(b_t, t) \dot{b}_t + \hat{V}'_t(b_t, t) + \frac{g_t}{\rho} \quad (50)$$

$$\begin{aligned} \text{s.t. } c_t &= A_t - \delta + d_t - g_t \\ \dot{b}_t &= r_t b_t + d_t - b_t g_t \\ b_t &\leq m_t \\ \dot{b}_t &\leq \bar{D} \end{aligned}$$

Guess that $\bar{V}(k_t, \tilde{b}_t, t) = \hat{V}(b_t, t) + \frac{\log k_t}{\rho}$. Define

$$b_t = \frac{\tilde{b}_t}{k_t} \quad c_t = \frac{\tilde{c}_t}{k_t}, \quad d_t = \frac{\tilde{d}_t}{k_t}, \quad g_t = \frac{\dot{k}_t}{k_t}$$

Notice that

$$\dot{b}_t = \frac{\dot{\tilde{b}}_t}{k_t} - b_t g_t$$

Then our guess implies

$$\bar{V}'_k(k_t, \tilde{b}_t, t) \dot{k}_t = -\hat{V}'_b(b_t, t) b_t g_t + \frac{g_t}{\rho}$$

$$\bar{V}'_b(k_t, \tilde{b}_t, t) \dot{\tilde{b}}_t = \hat{V}'_b(b_t, t) \frac{\dot{\tilde{b}}_t}{k_t}$$

From which,

$$\bar{V}'_k(k_t, \tilde{b}_t, t) \dot{k}_t + \bar{V}'_b(k_t, \tilde{b}_t, t) \dot{\tilde{b}}_t + \bar{V}'_t(k_t, \tilde{b}_t, t) = \hat{V}'_b(b_t, t) \dot{b}_t + \hat{V}'_t(b_t, t) + \frac{g_t}{\rho}$$

Substituting into the definition of $\bar{V}(k_t, \tilde{b}_t, t)$ we obtain

$$\rho \bar{V}(k_t, \tilde{b}_t, t) = \max_{g_t, d_t} u(c_t) + \log k_t + \hat{V}'_b(b_t, t) \dot{b}_t + \hat{V}'_t(b_t, t) + \frac{g_t}{\rho}$$

$$\begin{aligned}
\text{s.t. } c_t &= A_t - \delta + d_t - g_t \\
\dot{b}_t &= r_t b_t + d_t - b_t g_t \\
b_t &\leq m_t \\
\dot{b}_t &\leq \bar{D}
\end{aligned}$$

And we verify the guess

$$\bar{V}(k_t, \tilde{b}_t, t) = \hat{V}(b_t, t) + \frac{\log k_t}{\rho}$$

By noticing that $\bar{V}(1, b_t, t) = \hat{V}(b_t, t)$ we obtain the result.

For part *b*) and *c*) I define $\bar{D}(\tau, b_\tau, g_\tau) = -\rho b_\tau + g_\tau b_\tau + D$ for some constant $D > 0$. I show that the constraints of the type (10) are always binding. Consider any time after the turning point where $b_t < m_t$ and denote with $\bar{V}(b_t, t)$ the detrended value function $\bar{V}(1, b_t, t)$. Suppose by contradiction that (10) is not binding. Then,

$$\rho \bar{V}(b_t, t) = \max_{g_t, d_t} u(c_t) + \bar{V}'_b(b_t, t) \dot{b}_t + \bar{V}'_t(b_t, t) + \frac{g_t}{\rho} \quad (51)$$

$$\begin{aligned}
\text{s.t. } c_t &= A_t - \delta + d_t - g_t \\
\dot{b}_t &= \rho b_t + d_t - b_t g_t
\end{aligned}$$

The first order conditions, with respect to d and g are, respectively

$$\begin{aligned}
-\frac{1}{c_t} &= \bar{V}'_b \\
\frac{1}{c_t} &= \frac{1}{\rho} - b_t \bar{V}'_b
\end{aligned}$$

which together imply

$$c_t = \rho(1 - b_t) \quad (52)$$

and

$$\bar{V}'_b(b_t, t) = -\frac{1}{\rho(1 - b_t)} \Rightarrow \bar{V}''_{t,b}(b_t, t) = 0 \quad (53)$$

where $V''_{t,b}$ is the cross derivative. Indicating with $\bar{V}''_b(b_t, t)$ is the second derivative of $\bar{V}(b_t, t)$ with respect to its first argument, it is easy to see that the equations above imply

$$\frac{\bar{V}'_b}{\bar{V}''_b} = (1 - b_t) \quad (54)$$

Consumption then satisfies

$$\rho(1 - b_t) = A_t - \delta + d_t - g_t = A_t - \delta + \dot{b}_t - \rho b_t - (1 - b_t)g_t$$

or

$$A - \delta - \rho = (1 - b_t)g - \dot{b}_t \quad (55)$$

The envelope of the value function, using (53), gives

$$\rho \bar{V}'_b = \bar{V}''_b \dot{b}_t + \bar{V}'_b(\rho - g_t)$$

and by using (54)

$$\dot{b}_t = g_t(1 - b_t)$$

Finally, substituting into (55)

$$A_t - \delta - \rho = 0$$

which doesn't hold by assumption and shows the contradiction. We conclude that (10) binds after the turning point, unless $b_t = m_t$. In particular, this implies that

$$-\frac{1}{c_t} \leq \bar{V}'_b$$

Moreover, since $\bar{D} = \rho b_t + d_t - b_t g_t$ it follows that $c_t = A_t - \delta + \bar{D} - \rho b_t - (1 - b_t)g_t$. The first condition with respect to g_t gives

$$\frac{1}{c_t} = \frac{1}{\rho(1 - b_t)}$$

and then

$$\frac{1}{\rho(1 - b_t)} \geq -\bar{V}'_b \quad (56)$$

Consider now the situation at any time t prior to the turning point. Again, denote with $V(b)$, V^d , $\bar{V}(b, 0)$, $\bar{V}^d(0)$ and $V^0(b)$ the detrended value functions before or at the turning point. Take $b_t < m_t$ and assume, by contradiction, that $\dot{b}_t < \bar{D}$. The detrended problem then gives

$$(\rho + \pi_0)V(b_t) = \max_{g_t, d_t} u(c_t) + \pi_0 V^0(b_t) + V'_b \dot{b}_t + \frac{g_t}{\rho}$$

$$\begin{aligned} \text{s.t. } \quad c_t &= A_0 - \delta + d_t - g_t \\ \dot{b}_t &= r_t b_t + d_t - b_t g_t \end{aligned}$$

Assume that $V^0(b_t) = \bar{V}(b_t, 0)$. Hence, there is no default if the turning point is $\tilde{t} = t$ and then $r_t = \rho$. Again, equations (52)-(55) hold. This time, the envelop condition gives

$$(\rho + \pi_t)V'_b = V''_b \dot{b}_t + V'_b(\rho - g) + \pi_t \bar{V}'_b$$

and substituting for (52)-(55)

$$\dot{b}_t = g_t(1 - b_t) + \pi_0(1 - b_t) \frac{V'_b(b_t, t) - \bar{V}'_b(b_t, t)}{V'_b(b_t, t)}$$

From the first order conditions we know that

$$V'(b_t, S_t) = -\frac{1}{\rho(1 - b_t)}$$

Assume for simplicity that $b_t < m(0)$. By (56) we obtain

$$\dot{b}_t \geq (1 - b_t)g$$

which substituted into (55) gives the following contradiction

$$A_0 - \delta - \rho \leq 0$$

Now take $V^0(b_t) = \bar{V}^d(t)$ which implies default if the turning point is $\tilde{t} = t$. The interest rate is $r_t = \rho + \pi_0$. The first order conditions are the usual ones, while the envelop condition is

$$(\rho + \pi_0)V'_b = V''_b \dot{b}_t + V'_b(\rho + \pi_0 - g_t)$$

again we have

$$\dot{b}_t = (1 - b_t)g_t \tag{57}$$

Consumption is

$$c_t = A_0 - \delta + \dot{b}_t - (\rho + \pi_0)b_t - (1 - b_t)g_t$$

and then

$$A_0 - \delta - \rho - \pi_0 b_t = 0$$

Guess that $\pi_0 = \eta G_0 = \eta \frac{A_0 - \delta - \rho}{1 - (1 - \eta)m_0}$. The the equation above is rewritten as

$$0 = A - 0 - \delta - \rho - \pi_0 b_t = (A_0 - \delta - \rho) \frac{1 - m_0 + \eta(m_0 - b_t)}{1 - (1 - \eta)m_0} > 0$$

which is a contradiction. Therefore, given our guess for G_0 , either $b_t = m_0$ or $\dot{b}_t = \bar{D}$. The initial condition for all the households is $b_0 = m_0$. We have just showed that our guess for G_0 implies that $b_t = m_0$ for all $t < \tilde{t}$, otherwise (10) would be binding and then $\dot{b}_t = \bar{D} > 0$, which is not feasible. Given $b_t = m_0$ and $\dot{b}_t = 0$ the first order condition for $g_t = g_0$ gives

$$g_0 = \frac{A_0 - \delta - \rho - \pi_0 m_0}{1 - m_0}$$

An equilibrium requires $g_0 = G_0$ and then $\pi_0 = \eta g_0$. Substituting we verify our guess. It is easy to see that, if $m_0 = m(0)$ then the risk premium π_0 disappears from the first order condition for g_0 and an equilibrium growth rate G_0 as stated in the Lemma is found by setting $\eta^* = 0$. □

Proof of Lemma 2

Proof. Step 1. The system of differential equations in implicit form is

$$\begin{bmatrix} 0 \\ \dot{A}_t \\ \dot{\phi}_t \end{bmatrix} = \begin{bmatrix} (\rho + \theta) \log(1 - m_t) + g_t - g_t^d \\ -a(A_t) A_t G_t \\ -\phi_t \left[\theta + (1 - \phi_t)(g_t - g_t^d) + \theta \phi_t \frac{m_t}{1 - m_t} \right] \end{bmatrix} \quad (58)$$

The function $a(A_t) > 0$ is given in Section ??, while

$$f_2(m_t, A_t) = (m_t - 1)[(\rho + \theta) \log(1 - m_t) + \min\{(1 - \xi)A_t, A_t - \delta - \rho\}] - m_t(A_t - \delta - \rho)$$

The system of differential equations in explicit form is

$$\begin{bmatrix} \dot{m}_t \\ \dot{A}_t \\ \dot{\phi}_t \end{bmatrix} = \begin{bmatrix} f_2(m_t, A_t) \\ -a(A_t) A_t G_t \\ -\phi_t \left[\theta + (1 - \phi_t)(g_t - g_t^d) + \theta \phi_t \frac{m_t}{1 - m_t} \right] \end{bmatrix} \quad (59)$$

The function $a(A_t) > 0$ is given in section ?. Moreover

$$\begin{aligned} g_t &= \frac{A_t - \delta - \rho + \dot{m}_t}{1 - m_t} \\ g_t^d &= \max\{\xi A_t - \delta - \rho, 0\} \\ G_t &= (1 - \phi_t)(g_t - g_t^d) + g_t^d + \phi_t \theta \frac{m_t}{1 - m_t} \end{aligned}$$

The system in explicit form is

$$\begin{bmatrix} \dot{m}_t \\ \dot{A}_t \\ \dot{\phi}_t \end{bmatrix} = \begin{bmatrix} f_2(m_t, A_t) \\ -a(A_t)A_tG_t \\ -\phi_t \left[\theta + (1 - \phi_t)(g_t - g_t^d) + \theta\phi_t \frac{m_t}{1-m_t} \right] \end{bmatrix}$$

where

$$f_2(m_t, A_t) = (m_t - 1)[(\rho + \theta) \log(1 - m_t) + \min\{(1 - \xi)A_t, A_t - \delta - \rho\}] - m_t(A_t - \delta - \rho)$$

Write the system as $\dot{y}_t = z(y_t)$ with $y_t = (m_t, A_t, \phi_t)$ and $z_n(y)$ is the n -th element in the vector $z(y)$. Any solution to the system of differential equations must have the following property.

Property 1. *Suppose that for some $\tau, \tau_1 \in [t, T]$, with $\tau < \tau_1$*

$$m(\tau) > 0, \quad A(\tau) > \delta + \rho, \quad m(\tau_1) < 0$$

then, for some $\tau_2 \in (\tau, T)$

$$m(\hat{\tau}) \begin{cases} > 0 & \hat{\tau} \in [\tau, \tau_2) \\ = 0 & \hat{\tau} = \tau_2 \\ < 0 & \hat{\tau} \in (\tau_2, T] \end{cases}$$

$$A(\hat{\tau}) > \delta + \rho \quad \forall \hat{\tau} \in [\tau, T]$$

The property is proven as follows. The existence of τ_2 where $m(\tau_2) = 0$ is a consequence of the continuity of $m(\cdot)$. Now suppose that $m(\hat{\tau}) > 0$ for $\hat{\tau} \in [\tau, \tau_2)$. We can show that $A(\tau_2) > \delta + \rho$. In fact, assume that $A(\tau_2) = \delta + \rho$. We then have

$$g^d(\tau_2) = 0$$

$$g(\tau_2) - g^d(\tau_2) = g(\tau_2) = 0$$

and thus

$$\dot{m}(\tau_2) = 0, \quad G(\tau_2) = 0$$

so that $m(\hat{\tau}) = 0$, $A(\tau_2) = \delta + \rho$ for all $\hat{\tau} \in [\tau_2, T]$, which is a contradiction. Suppose, instead, that $A(\tau_2) < \delta + \rho$. Then,

$$\dot{m}(\tau_2) > 0$$

again a contraction with $m(\tau_2) = 0$. We conclude that $A(\tau_2) > \delta + \rho$ and thus

$$\dot{m}(\tau_2) < 0$$

Notice that, as long as $A(\hat{\tau}) > \delta + \rho$ we have $\dot{m}(\hat{\tau}) < 0$. Suppose that, at some τ_3 , we have $A(\hat{\tau}_3) = \rho + \delta$ and $A(\hat{\tau}) > \rho + \delta$ for $\hat{\tau} \in [\tau_2, \tau_3)$. Then, since $m(\tau_3) < 0$ and

$$\begin{aligned} g^d(\tau_3) &= 0 \\ g(\tau_3) - g^d(\tau_3) &= g(\tau_3) < 0 \end{aligned}$$

it follows that

$$G(\tau_3) < 0$$

which is a contradiction with $A(\hat{\tau}_3) = \rho + \delta$. Therefore we must have

$$m(\tau) < 0 \quad \tau \in (\tau_2, T]$$

and we have established the results of Property 1.

We now move to show the existence of a global solution to the system (59). Define the set Y as

$$Y = \{(m, A, \phi) | 0 \leq m \leq m_2, 0 \leq A \leq A_0, 0 \leq \phi \leq 1\}$$

where m_2 is given in (46). For $p > 0$ and small, denote with $B(y; p)$ the open ball centered at y with radius p and define \bar{Y} as

$$\bar{Y} = \bigcup_{y \in Y} B(y, p)$$

It is easy to show that there exists a constant $L > 0$

$$|z_n(y)| \leq L \quad \forall y \in \bar{Y}$$

with $n = 1, 2, 3$. Moreover, call $J_{i,j}(y)$ the typical element of the the Jacobian matrix of $z(y)$ at any differentiable point y . We can show that there is a constant $L_1 > 0$ such that

$$|J_{i,j}(y)| \leq L_1 \quad \forall y \in \bar{Y}$$

$i = 1, 2, 3, j = 1, 2, 3$. Then the following Lipschitz condition holds

$$\sum_n |z_n(y) - z_n(x)| \leq L_1 \sum_n |y_n - x_n|$$

$y, x \in \bar{Y}$. Define T_1 as

$$T_1 = \frac{p}{L}$$

General arguments on the existence of local solutions⁸ to the system (59) imply that, for any $y^0 \in Y$ there exists one and only one continuous solution $y(\tau)$ to the system (59), for $0 \leq \tau \leq T_1$ and

$$y(0) = y^0$$

$$|y_n(\tau) - y_n(0)| \leq p \quad 0 \leq \tau \leq T_1$$

$n = 1, 2, 3$. The existence and uniqueness result show the existence of a continuous function $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that, if $y(\tau)$ is a solution to the system (59) for $0 \leq \tau \leq T_1$, then

$$y(T) = F(y(t))$$

Define the set M_0 as

$$M_0 = [0, m_2]$$

Consider the two initial states $\underline{y}_0 = (0, A_0, \phi_0)$ and $\bar{y}_0 = (m_2, A_0, \phi_0)$, with $\phi_0 \in \{0, 1\}$ to represent the case of default and of no default at the turning point $t = 0$. Notice that $\underline{y}_0, \bar{y}_0 \in Y$ and that $\underline{m}_0(0) < 0$, $\bar{m}_0(0) \geq 0$ which implies

$$\underline{m}_0(T_1) = F_1(\underline{y}_0) < 0$$

$$\bar{m}_0(T_1) = F_1(\bar{y}_0) > m_2$$

The first conclusion is drawn from Property 1, while the second is due to the observation, and the discussion, of the phase diagram in Figure 4. By continuity of $F(\cdot)$, there exists an interval $M_1 = [\underline{m}_1, \bar{m}_1] \subseteq M_0$ such that

$$F_1(\underline{m}_1, A_0, \phi_0) = 0$$

$$F_1(\bar{m}_1, A_0, \phi_0) = m_2$$

$$0 < F_1(m, A_0, \phi_0) < m_2 \quad \forall \underline{m}_1 < m < \bar{m}_1$$

For $\tau \in [0, T_1]$ let us call $\underline{y}_1(\tau) = (\underline{m}_1(\tau), \underline{A}_1(\tau), \underline{\phi}_1(\tau))$ the solution with initial condition $\underline{y}_1(0) = (\underline{m}_1, A_0, \phi_0)$. By Property 1 we have that $F_2(\underline{y}_1(0)) \geq \rho + \delta$. Then, for all $\tau \in [0, T_1]$

$$\underline{m}_1(\tau) \geq 0$$

⁸See, for instance, Franklin (1954).

$$\underline{A}_1 \geq 0$$

which imply

$$\begin{aligned} \underline{m}_1(\tau) &\in [0, m_2) \\ \underline{A}_1(\tau) &\in [\rho + \delta, A_0] \end{aligned} \tag{60}$$

for $\tau \in (0, T_1)$.

If $F_2(\underline{y}_1(0)) = \rho + \delta$ then the functions $\underline{m}_1(\tau)$ and $\underline{A}_0(\tau)$ are constant from time T_1 on and, because of (60), we have found a solution to the system (59) with the required restrictions on m , A and ϕ .

If $F_2(\underline{y}_1(0)) > \rho + \delta$ repeat the procedure above as follows. Define the set Y_{T_1} as

$$Y_{T_1} = \{F(m, A_0, \phi_0) | m \in M_1\}$$

Notice that

$$Y_1 \subseteq Y$$

Define $T_2 = T_1 + \frac{p}{L}$. We then know that, for any $y_{T_1} \in Y_{T_1}$ there exists a unique solution $y(\tau)$ to (59) for $\tau \in [T_1, T_2]$ with $y(T_1) = y_{T_1}$. For any $y_{T_1} \in Y_{T_1}$ we can write

$$y(T_2) = F(y_{T_1}) = F(F(y_0))$$

with $y_0 = (m, A_0, \phi_0)$ and for one and only one $m \in M_1$. Again, notice that

$$F_1(F(\underline{m}_1)) < 0$$

$$F_1(F(\bar{m}_1)) > m_2$$

Hence, using the continuity of $F(F(\cdot))$, we construct a new interval $M_2 = [\underline{m}_2, \bar{m}_2]$ with the usual properties. If $F_2(F(\underline{y}_2)) = \rho + \delta$ then we have found a solution to the system (refsystem) with initial condition \underline{y}_2 . Otherwise, if $F_2(F(\underline{y}_2)) > \rho + \delta$, we construct a new set $M_3 \subseteq M_2$ and repeat the process. It could be the case that the procedure never stops, and we have then to construct an infinite sequence of nested closed intervals M_1, M_2, \dots . For $n = 1, 2, \dots$ define the restrictions $\bar{M}_n \subseteq M_n$ in the following way. Consider any $m \in M_n$ and an initial condition (m, A_0, ϕ_0) . Call $y(\tau) = (m(\tau), A(\tau), \phi(\tau))$ the unique solution with $m(0) = m$ for $\tau \in [0, T_n]$. Then $m \in \bar{M}_n$ if and only if the following holds

$$\begin{aligned} m(\tau) &\in [0, m_2] & \tau &\in [0, T_n] \\ A(\tau) &\in [\rho + \delta, A_0] & \tau &\in [0, T_n] \end{aligned} \tag{61}$$

Because of (60) the sets \bar{M}_n are non empty, since they always contain \underline{m}_n . By continuity of $y(\tau)$ in $m(0)$ the sets \bar{M}_n are closed. Moreover, it is also easy to see that $M_0 \supseteq \bar{M}_1 \supseteq \bar{M}_2 \supseteq \dots$. Since M_0 is compact and \bar{M}_n is an infinite sequence of closed subsets of M_0 having the finite intersection property we conclude that

$$\bar{M} \equiv \bigcap_{n=1}^{\infty} \bar{M}_n \neq \emptyset$$

The set \bar{M} is the set defining initial conditions $m(0)$ to which we can associate a unique solution to the system of differential equations.

Step 2.

The part of the Lemma requiring that any $m(0)$ is arbitrarily close to m_2 as $\gamma \rightarrow 0$ is proven as follows. Take any initial condition $\tilde{y} = (m, A_0, \phi_0)$ for an arbitrary $m < m_2$. I show that, as γ becomes small, the initial condition $m_\gamma(0)$ of any solution $y_\gamma(\tau)$ must lie in the interval (m, m_2) .

Define $y_0(\tau) = (m^0(\tau), A^0(\tau), \phi^0(\tau))$ the solution to (59) corresponding to $\gamma = 0$ and initial condition $y_0 = \tilde{y}$, for $\tau \in [0, T]$ and $T > 0$. The existence of a solution $m(\tau)$ for $\tau \in [0, T]$ and T not too large is guaranteed by the arguments presented above. Notice in particular that, from (??), $\dot{w}(\tau) = 0$ is a solution for all τ , hence

$$A^0(\tau) = A_0 \quad \forall \tau \in [0, T]$$

Moreover, since $m < m_2$,

$$\dot{m}^0(\tau) < 0 \quad \forall \tau$$

It is easy to prove that, given two initial conditions $m > \hat{m}$, the respective solutions $m^0(\tau)$ and $\hat{m}^0(\tau)$ satisfy

$$m^0(\tau) > \hat{m}^0(\tau) \quad \tau \in [0, T]$$

Given the initial condition $m^0(0) = m$ we can then choose a time T

$$m^0(T; m) = -\epsilon(m)$$

$\epsilon(m) > 0$, where $\epsilon(m)$ is decreasing in m . Also, since a solution for $\tau \in [0, T]$ exists if $\gamma = 0$, then a solution $y_\gamma(\tau)$ with $y_\gamma(0) = \tilde{y}$ must exist when $\gamma > 0$ and small. The function $y_\gamma(\cdot)$ is continuous in γ with respect to the sup norm, i.e. for any $\epsilon_1 > 0$ there exists an $\epsilon_2 > 0$ such that

$$\sup_{\tau \in [0, T]} \|y_\gamma(\tau) - y_0(\tau)\| < \epsilon_1 \quad \forall \gamma \in [0, \epsilon_2]$$

Now take $\epsilon_1 < \epsilon$ and suppose that, for some $\gamma < \epsilon_2$, there is a global solution $y_\gamma(\tau)$ to (59) for $\tau \geq 0$ such that $y_\gamma(0) = \tilde{y}$. Then $y_\gamma(\tau)$ is the unique solution to (59) for $\tau \in [0, T]$. Continuity implies

$$|m_\gamma(T) - m^0(T)| < \epsilon_1 \Rightarrow m_\gamma(T) < \epsilon_1 - \epsilon < 0$$

We conclude that, given an $m < m_2$ we can always find a γ sufficiently small so that $m_\gamma(0) = m$ cannot be the initial condition of a solution to (59) with the required restrictions on m . In particular, any solution $m_\gamma(\tau)$ must satisfy

$$m < m_\gamma(0) < m_2$$

Hence, as γ gets close to zero, any solution is such that the initial condition $m_\gamma(0)$ gets close to m_2 .

Step 3.

This part proves uniqueness for an initial condition $\phi_0 = 0$. Consider the following property.

Property 2. *Suppose that there are two solutions $y(\tau)$ and $\hat{y}(\tau)$ corresponding to initial conditions $y(0) = (m, A_0, 0)$ and $\hat{y}(0) = (\hat{m}, A_0, 0)$ such that $m > \hat{m}$. Then*

$$m(\tau) \geq \hat{m}(\tau) \quad \tau \geq 0$$

$$A(\tau) < \hat{A}(\tau) \quad \tau > 0$$

In fact, it is easy to see that $m > \hat{m}$ implies $G(0) > \hat{G}(0)$ so that $A(\tau) < \hat{A}(\tau)$ and $m(\tau) > \hat{m}(\tau)$ in a neighborhood of $\tau = 0$. Suppose that there exists at a certain time τ_1 when $A(\tau_1) = \hat{A}(\tau_1)$, $m(\tau) > \hat{m}(\tau_1)$ and $A(\tau) < \hat{A}(\tau)$ for $\tau \in [0, \tau_1)$. Then it must be the case that

$$g(\tau_1) - g^d(\tau_1) > \hat{g}(\tau_1) - \hat{g}^d(\tau_1)$$

But since $g^d(\tau_1) = \hat{g}^d(\tau_1)$,

$$g(\tau_1) > \hat{g}(\tau_1) \Rightarrow G(\tau_1) > \hat{G}(\tau_1)$$

which is a contradiction with $A(\tau_1) = \hat{A}(\tau_1)$. Now suppose that there exists at a certain time τ_1 when $m(\tau_1) = \hat{m}(\tau_1)$, $A(\tau) < \hat{A}(\tau_1)$ and $m(\tau) > \hat{m}(\tau)$ for $\tau \in [0, \tau_1)$. Then it must be the case that

$$g(\tau_1) - g^d(\tau_1) = \hat{g}(\tau_1) - \hat{g}^d(\tau_1)$$

But since $A(\tau_1) \leq \hat{A}(\tau_1)$ and $m(\tau_1) = \hat{m}(\tau_1)$,

$$\dot{m}(\tau_1) > \dot{\hat{m}}(\tau_1)$$

which is again a contradiction and we establish the results of the property. If $m(\tau)$ and $\hat{m}(\tau)$ are both solutions then $A(\tau) - \hat{A}(\tau) \rightarrow 0$ since the marginal product of capital converges always to $\rho + \delta$. Consequently, aggregate capital $K(\tau) - \hat{K}(\tau) \rightarrow 0$. Take any time τ sufficiently large so that $g^d(\tau) = \hat{g}^d(\tau) = 0$. Since, by the previous property, $m(\tau) > \hat{m}(\tau)$

$$g(\tau) > \hat{g}(\tau)$$

Again, by the previous property

$$A(\tau) < \hat{A}(\tau) \Rightarrow K(\tau) > \hat{K}(\tau)$$

But then

$$\lim_{\tau \rightarrow \infty} K(\tau) - \hat{K}(\tau) > 0$$

which is a contradiction. We conclude that, if $\phi(0) = 0$ the solution is unique. \square

Proof of Proposition 1

Proof. If $M_1 = \emptyset$ the proposed leverage limits exist, are unique sustainable by Lemma 2. The proposed value of m_0 is the largest for an equilibrium with no default by construction. I show that there is no equilibrium with default. Since γ is small, any $\bar{m}(0)$ is arbitrarily close to m_2 . Any equilibrium with default would then require $m_0 > m_2$. But this is not possible since $M_1 = \emptyset$ and this case corresponds to the dashed line in Figure 5.

Assume that $M \neq \emptyset$. For the proposed equilibrium $r_0 = \rho + \eta\pi_0$. Given an initial condition $m_0 = \max\{M_1\}$ for the leverage ratio, the household might have an incentive to default and then choose a leverage ratio $m'_0 = \bar{m}(0)$, in order to pay a lower interest rate $r_0 = \rho$. I show that this is not the case. Note that $\bar{V}^0(m'_0) = \bar{V}(m_0) = \bar{V}^d = \bar{V}^0(m_0)$. Moreover, since $\bar{m}(0)$ is arbitrarily close to m_2 ,

$$\begin{aligned} \rho(\rho + \pi'_0)[V(m_0) - V(m'_0)] &= [\rho \log(1 - m_0) + G_0] - [\rho \log(1 - m'_0) - g'_0] \\ &= [g^d - \theta \log(1 - m_0)] - [g^d - \theta \log(1 - m'_0)] + \varepsilon_\gamma \\ &= \theta \log \frac{1 - m'_0}{1 - m_0} + \varepsilon_\gamma > 0 \end{aligned}$$

The inequality holds because m_0 and m_2 don't depend on γ , $m_0 > m_2 \geq m'_0$ and ε_γ can be made arbitrarily small. Choosing m_0 delivers then more utility to the household. This last part shows also that, for γ small $g'_0 < G_0$. Finally, I prove that if $M \neq \emptyset$ there is no equilibrium with endogenous leverage limits that excludes default at the turning point. Suppose, by contradiction, that such an equilibrium exists. The initial condition would be $m'_0 = \bar{m}(0)$. As usual define $m_0 = \max\{M_1\}$. Define G_0 the aggregate growth rate when the capital of all the households grow with the leverage ratio m_0 and an interest rate $r_0 = \rho + \eta G_0$. Define G'_0 the aggregate growth rate when the capital of all the households grow with the leverage ratio m'_0 and an interest rate $r'_0 = \rho$. Define g_0 the growth rate of the capital of a single (deviating) household with a leverage ratio m_0 and an interest rate $r''_0 = \rho + \eta G'$. We know that $G_0 > G'_0$. Moreover $g_0 > G_0$, since $r''_0 < r_0$. Then

$$\begin{aligned} \rho(\rho + \pi_0)[V(m_0) - V(m'_0)] &= [\rho \log(1 - m_0) + g_0] - [\rho \log(1 - m'_0) + g'_0] \\ &> [\rho \log(1 - m_0) + G_0] - [\rho \log(1 - m'_0) - g'_0] \\ &= \theta \log \frac{1 - m'_0}{1 - m_0} + \varepsilon_\gamma > 0 \end{aligned}$$

If, starting with an initial condition m'_0 the household makes the leverage ratio jump to m_0 and remain there until the turning point, she would receive a value \tilde{V}

$$\tilde{V} = V(m_0) + \frac{1}{\rho} \log \frac{1 - m'_0}{1 - m_0} > V(m'_0)$$

The household would then choose a leverage ratio m_0 , which is strictly larger than m'_0 and supportable by construction. Therefore the leverage bound m' is not endogenous.

For the last part notice that, for any m , $f_1(m; \mathcal{E}^l) \geq f_1(m; \mathcal{E})$. □

6 Appendix B

I derive the equations determining the evolution of the aggregate growth rate of capital and of the fraction ϕ of capital allocated to households in default.

Total capital K evolves according to

$$K_{t+\epsilon} = (1-\phi_t)K_t e^{\int_0^\epsilon g_{t+\tau} d\tau} + \phi_t K_t \left[e^{-\theta\epsilon} e^{\int_0^\epsilon g_{t+\tau}^d d\tau} + \int_0^\epsilon \frac{e^{\int_0^\tau g_{t+T}^d dT + \int_\tau^\epsilon g_{t+T} dT}}{1-m_{t+\tau}} \theta e^{-\theta\tau} d\tau \right]$$

which gives

$$K_{t+\epsilon} = K_t \left[(1-\phi_t)(1+g_t\epsilon) + \phi_t \left[1 + (g_t^d - \theta)\epsilon + \frac{\theta}{1-m_t}\epsilon \right] \right] + o(\epsilon)$$

Define $G_t = \lim_{\epsilon \rightarrow 0} \frac{K_{t+\epsilon} - K_t}{\epsilon K_t}$ and equation (42) is obtained.

The evolution of the share ϕ_t is the following,

$$1 - \phi_{t+\epsilon} = \frac{K_{t+\epsilon} - \phi_t K_t e^{-\theta\epsilon} e^{\int_0^\epsilon g_{t+\tau}^d d\tau}}{K_{t+\epsilon}}$$

$$\phi_{t+\epsilon} = \phi_t \frac{e^{-\theta\epsilon + \int_0^\epsilon g_{t+\tau}^d d\tau}}{e^{\int_0^\epsilon G_{t+\tau} d\tau}}$$

$$\phi_{t+\epsilon} = \phi_t [1 + (g_t^d - \theta - G_t)\epsilon] + o(\epsilon)$$

from which we get (43).